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ROBINSON'S MATHEMATICAL SERIES.

A NEW TREATISE

ON THE ELEMENTS OF

THE DIFFERENTIAL AND INTEGRAL

CALCULUS.

EDITED BY

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PREFACE.

THE design in preparing this treatise on the Differential and Integral Calculus has been, not so much to produce a work that should cover the whole ground of this extensive and rapidly extending branch of mathematics, as to produce one that should be complete within the limits assigned it, and adapted to the wants of students in the higher schools and colleges of this country. Many of the subjects are much more fully discussed in this volume than in other elementary treatises; while many are entirely omitted here which are generally included in such works, though they are not essential to, and are rarely embraced in, the college course in this or in other countries. The necessity devolved on the author, either to be limited in the number and full in the treatment of the subjects selected, or full in the number of subjects, and limited in their discussion. The former choice was taken, keeping in view the logical and progressive development of the principles.

This will account for the omission, among other subjects, of the integration of differential equations of the different orders, and of the "Calculus of Variations," the latter of which, when fully treated, would make a volume equal to the present in size.

It will be found, however, that the time usually given to this study will render it impossible to take, in course, all the subjects herein treated. The following are what may be left out in the class-room without serious breaks in continuity: —

Differential Calculus. — *Part First.* — Section V., from Article 68 to the end of the Section. The whole of Section VII. Section X., from Article 110 to the end of the Section. The whole of Section XII. Section XIV., from Article 139 to the end of the Section.

Differential Calculus. — *Part Second.* — The whole of Section II. Section IV., from Article 177 to Article 181; from Article 188 to the end of the Section.

Integral Calculus. — From Example 4, Section IV., to the end of the Section. The whole of Sections IX. and X.

It will be observed that the fundamental proposition of the Differential Calculus is based on the doctrine of limits; and that of the Integral Calculus, on that of the summation of an infinite series of infinitely small terms. The author adopts these methods merely on logical grounds, but ventures the opinion that these, and what are called the infinitesimal methods, are based on the same metaphysical principles.

THE AUTHOR.

November, 1867.

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DIFFERENTIAL CALCULUS.

PART FIRST.

SECTION I.

GENERAL PRINCIPLES AND DEFINITIONS.

1. In the branch of mathematics of which it is now proposed to treat, we have to deal with two classes of quantities, —constants and variables: constants, which undergo no change of value in the investigations in which they are involved; variables, which may pass through all values within limits that may be restricted or indefinite.

Variables are usually represented by the final letters of the Roman alphabet; and constants, by the first letters of this, and sometimes, also, of the Greek alphabet.

2. When variable quantities are so connected, that, one or more of them being given, the values of the others become fixed, the latter are said to be functions of the former, which are called the independent variables, or simply the variables. The functions are also called *dependent* variables.

Thus, in the equation

$$y = ax^2 + bx + c,$$

y is a function of x, and in this case becomes not only fixed,

but known, so soon as a value is assigned to x. So also, in the equation

$$y = ax^2 + bx + cz + d,$$

y is a function of the two variables x and z, and is known in value when values are given to x and z.

3. An ***Explicit Function*** is one in which the dependent variable is given directly in terms of those which are regarded as independent.

In the examples given above, y is an explicit function of x in the first, and of x and z in the second. In general reasoning, when we are not concerned with the particular form of the function, explicit functions are denoted by the symbols

$$y = F(x),\ y = f(x),\ y = \varphi(x, z), \text{ \&c.}$$

4. An ***Implicit Function*** is one in which the relation between the function and the independent variable or variables is expressed by an equation that has not been resolved in respect to the function.

Thus
$$ax + by + c = 0,$$
$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$
$$x^3 - az^2 + cy^2xz + d = 0,$$

are equations which require solution to render the variable, taken as dependent, an explicit function of the independent variables. Such functions are also designated by the symbols

$$F(x, y) = 0,\ \varphi(x, y, z) = 0, \text{ \&c.}$$

5. Functions are also classified, in reference to their composition, into *simple* or *compound*, according as they are the result of one or of several operations performed on the variables. They are algebraic, when, in the construction of the

function, the only operations to which the variables are subjected are those of addition, subtraction, multiplication, division, involution denoted by constant exponents, and evolution denoted by constant indices; transcendental, when, in the composition of the function, the variables have been subjected to other operations, combined or not with those regarded as algebraic.

Thus $y = a^x$, $y = \log. x$, $y = \sin. x$, $y = \sin.^{-1} x$,*

are examples of transcendental functions, and are exponential, logarithmic, or circular, depending on the mode in which the variable enters the functions.

6. A function may be continuous or discontinuous. It is continuous, when, by causing the variable to pass gradually from any value to another separated from the first by a finite interval through all the intermediate states of value, the function will itself pass gradually through all the values intermediate to those corresponding to the extreme values of the variable; and when, besides, the law of dependence of the function upon the variable does not change abruptly in the interval.

$y = F(x)$ is continuous, if, by giving to x the infinitely small increment $h = \Delta x$, y receives the infinitely small increment $\Delta y = F(x + \Delta x) - F(x)$. When the law of the function is such that these conditions are not satisfied, the function is discontinuous.

7. The ***Limit of a Function*** is the value towards which it converges, and from which it finally differs by less than any assignable value, when the variable upon which it depends itself converges towards some fixed value.

* Read arc whose sine is x, and frequently written arc ($\sin. = x$). The notations $\cos.^{-1} x$, $\tan.^{-1} x$, &c., have like significations.

It is of the highest importance that we should have a clear conception of the nature of limits as above defined, as this conception is at the foundation of the differential calculus as developed in the following pages. The following examples will illustrate the meaning of *limit*, and give distinct notions on the subject to those who have not already formed them:—

1st, In the geometrical series

$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\&c.,$$

the sum S of the first n terms is given by the formula

$$S=\frac{a(1-r^n)}{1-r}=\frac{\frac{1}{2}\left(1-(\frac{1}{2})^n\right)}{1-\frac{1}{2}}=1-(\tfrac{1}{2})^n;$$

and it is obvious, that, as n increases, $(\frac{1}{2})^n$ decreases; and, when n becomes greater than any assignable quantity, $(\frac{1}{2})^n$ becomes less than any assignable quantity. In the language of the definition, as n converges towards infinity, S converges towards unity. Hence the limit of the sum of this series, when n is indefinitely increased, is 1.

2d, The ratio of an arc of a circle to its sine has unity for its limit when the arc converges to zero; that is, limit $\frac{\sin. x}{x}=1$. For it is plain in the first place, that, for sensible values of x, the sine is less than the arc. And again: since the triangle formed by the radius, the tangent, and the secant, has for its measure $\frac{1}{2}$ $R.$ tan. x, while the corresponding sector is measured by $\frac{1}{2}$ $R.\,x$, it follows that the arc x is less than tan. x.

Therefore

$$\frac{\sin. x}{x}<\frac{x}{x}=1,\quad \frac{\sin. x}{x}>\frac{\sin. x}{\tan. x}=\cos. x.$$

Hence we conclude, that, for sensible values of the arc x, $\frac{\sin. x}{x}$

is always included between two ratios, both of which have unity for their limit. It must, then, have the same limit; and we have $\lim. \frac{\sin. x}{x} = 1$.

Again:

$$\frac{\sin. x}{x} = \frac{\frac{\sin. x}{\cos. x} \cos. x}{x} = \frac{\tan. x}{x} \cos. x, \text{ and}$$

$$\lim. \frac{\sin. x}{x} = 1 = \lim. \frac{\tan. x}{x} \cos. x; \text{ but } \lim. \cos. x = 1,$$

therefore the limit of $\frac{\tan. x}{x}$ must also be unity; *i.e.*, the limit of the ratio of an arc to its tangent is unity.

Cor. The limiting ratio of the arc to its sine, and of the arc to its tangent, each being unity, it follows, that, when the arc is infinitesimal, the arc and its sine, and the arc and its tangent, may be regarded as equal.

3d, For another example, let us take $y = \frac{x}{1+x}$, and trace out the series of values which y assumes when positive values are given to x. Beginning with $x = 0$, we have $y = 0$. By division, the value of y takes the form $1 - \frac{1}{x+1}$; from which it is seen, that, as x increases, the subtractive part of the value of y decreases, and y itself increases; and, as x approaches $+\infty$, y approaches its limit 1: but, for all finite positive values of x, the values of y are less than 1. The difference between y and this limit can be made as small as we please by giving to x a value sufficiently great. Thus, if we wish to make this difference less than $\frac{1}{1,000,000}$, we make $x = 1,000,000$.

In this same example, let us now give to x negative values, and observe the changes in the value of y as x increases nega-

tively from 0, and approaches $-\infty$. Replace x by $-t$, then

$$y=\frac{x}{x+1}=\frac{-t}{-t+1}=\frac{t}{t-1},$$

and let us consider the values of y answering to values of t between the limits 0 and 1. Beginning with $t=0$, we have $y=0$: for all other values of t between these limits, the denominator of y being negative, y is itself negative. As t increases, y increases numerically; and, when t differs from unity by less than any assignable quantity, y is greater numerically than any assignable quantity; that is, $-\infty$ is the limit of y for $t=1$, which answers to $x=-1$. This is equivalent to saying that y has then no finite limit.

When t passes 1, the denominator of the fraction $\frac{t}{t-1}$ becomes positive, and y changes from negative to positive. In this case, y passes abruptly from $-\infty$ to $+\infty$ while t is passing through the value 1. The value of y may now be put under the form $y=\dfrac{1}{1-\frac{1}{t}}$. For all finite values of t greater than $+1$, y is greater than 1: but y decreases as t increases; and finally, when t becomes greater than any assignable quantity, y will differ from its limit, unity, by less than any assignable quantity.

Trigonometry furnishes a case of limit similar to that of this example, when t passes through the value unity. As an arc increases continuously from 0, its tangent also increases continuously, but more rapidly than the arc; and, as the arc approaches 90°, the tangent approaches its indefinite limit $+\infty$. When the arc passes through the value 90°, the tangent changes suddenly from an indefinitely great positive to an indefinitely great negative quantity.

8. The exact meaning of the word "limit" will be understood from what precedes; but it is well to call attention to abbreviations of expression frequently used in this connection. In finding the limit of $\frac{\sin. x}{x}$ when x is diminished without limit, it would be said,

$$\text{limit} \frac{\sin. x}{x} = 1 \text{ when } x = 0;$$

but it must be borne in mind that $\frac{\sin. x}{x}$ cannot reach this limit so long as x has any value. And, if we actually make $x = 0$, the ratio $\frac{\sin. x}{x}$ has no meaning; in fact, ceases to exist. It is true, that if x be not supposed to vanish, but simply to differ from 0 by less than any assignable quantity, that is, if x becomes infinitesimal, the ratio retains its significance, and its value will differ from its limit unity by less than any assignable quantity.

In this case, the language is an abbreviation for this or its equivalent: "As x is diminished, the ratio $\frac{\sin. x}{x}$ converges towards unity, and can be made to differ from it by as small a quantity as we please by taking x sufficiently near zero." And, in all similar cases, the language is to be interpreted in the same way.

In other cases of limits, the inconsistency just pointed out does not present itself. Any finite value of y, in the example $y = \frac{x}{x+1}$, that answers to an assumed and finite value of x, may be taken as a limit of y; and it would be strictly correct to say

$$\text{limit of} \frac{x}{x+1} = \frac{1}{2} \text{ when } x = 1.$$

This corresponds to the definition of limit given in Art. 7.

9. Rules for the evaluation of functions, which, for particular values of the variable, assume the indeterminate forms $\frac{0}{0}$, $\pm\frac{\infty}{\infty}$, $0 \times \infty$, $0^0 \pm 1^\infty$, will be established in a subsequent section: but it is necessary for our purposes to consider in this place the function $y = (1 + x)^{\frac{1}{x}}$, and to find its limiting value when $x = 0$; the function then taking the indeterminate form 1^∞.

The variable x may converge towards its assigned limit zero through either positive or negative values. Let us first suppose x to be positive, and represent it by the fraction $\frac{1}{m}$; then, as x diminishes, m increases; and, when x becomes a very small quantity, m becomes a very great quantity.

If m be an entire positive number, we have, by the Binomial Formula,

$$\left(1 + x\right)^{\frac{1}{x}} = \left(1 + \frac{1}{m}\right)^m = 1 + 1 + \frac{m}{1}\frac{m-1}{2}\frac{1}{m^2}$$
$$+ \frac{m}{1}\frac{m-1}{2}\frac{m-2}{3}\frac{1}{m^3} + \&c.,$$

a development which will contain $m + 1$ terms.

Dividing both numerator and denominator of each term by the power of m that enters the denominator, we find

$$\left(1+x\right)^{\frac{1}{x}} = \left(1+\frac{1}{m}\right)^m = 2 + \frac{1}{2}\left(1-\frac{1}{m}\right) + \frac{1}{2}\frac{1}{3}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)$$
$$+ \frac{1}{2}\frac{1}{3}\frac{1}{4}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\left(1-\frac{3}{m}\right) + \&c.$$

Under the hypothesis that m is a positive whole number, the expressions $1 - \frac{1}{m}$, $1 - \frac{2}{m}$, $1 - \frac{3}{m}$, &c., will each be positive,

and less than unity. Therefore $\left(1+\frac{1}{m}\right)^m = 2 +$ some positive quantity; that is

$$\left(1+\frac{1}{m}\right)^m > 2.$$

Again: the development will be increased in value both by neglecting the subtractive terms $\frac{1}{m}$, $\frac{2}{m}$, $\frac{3}{m}$, &c., and also by replacing each of the denominators 2, 3, 4, &c., by the least denominator 2; that is, the true value of the development is less than 2 plus the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{\&c.}$$

But this series cannot exceed 1, however far continued: therefore $\left(1+x\right)^{\frac{1}{x}} = \left(1+\frac{1}{m}\right)^m$ is always included between the limits 2 and 3.

If $\frac{1}{x} = m$ is a fractional number, it will be found between two consecutive whole numbers m and $n = m + 1$. Let s and t be two positive proper fractions, whose sum is always equal to 1, and make

$$\frac{1}{x} = m + s = n - t, \text{ whence } \begin{matrix} x < \frac{1}{m}, \\ x > \frac{1}{n} \end{matrix}$$

and the expression $\left(1+x\right)^{\frac{1}{x}}$ will be included between

$$\left(1+\frac{1}{m}\right)^{\frac{1}{x}} = \left(1+\frac{1}{m}\right)^{m+s} = \left\{\left(1+\frac{1}{m}\right)^m\right\}^{1+\frac{s}{m}}$$

and

$$\left(1+\frac{1}{n}\right)^{\frac{1}{x}} = \left(1+\frac{1}{n}\right)^{n-t} = \left\{\left(1+\frac{1}{n}\right)^n\right\}^{1-\frac{t}{n}}$$

Now, as x decreases indefinitely, m and n increase to infinity, and the two quantities $\left(1+\frac{1}{m}\right)^m$ and $\left(1+\frac{1}{n}\right)^n$ both converge to the same limit, which, as was proved above, is included between 2 and 3; while the exponents $1+\frac{s}{m}$, $1-\frac{t}{n}$, converge to the limit 1.

It follows, therefore, that the two expressions $\left(1+\frac{1}{m}\right)^{\frac{1}{x}}$, $\left(1+\frac{1}{n}\right)^{\frac{1}{x}}$, have the same limit, and that this limit is the same as that of $\left(1+\frac{1}{m}\right)^m$ when m, regarded as a positive whole number, is indefinitely increased.

Finally, if x is negative, and either entire or fractional, make $x=-\frac{1}{z+1}$: so that, for all values of x numerically greater than 1, z must be negative and included between the limits 0 and 1; but, for values of x less than 1 (and it is with these alone that we are now concerned), z must be positive, and increase to infinity as x decreases to zero. Making this substitution for x, we have

$$\left(1+x\right)^{\frac{1}{x}}=\left(1-\frac{1}{z+1}\right)^{-(z+1)}=\left(\frac{z+1}{z}\right)^{z+1}$$

$$=\left(1+\frac{1}{z}\right)^{z+1}=\left\{\left(1+\frac{1}{z}\right)^z\right\}^{1+\frac{1}{z}}$$

Hence, when x approaches its limit zero through negative values, the limit of the expression $(1+x)^{\frac{1}{x}}$ = limit $\left\{\left(1+\frac{1}{z}\right)^z\right\}^{1+\frac{1}{z}}$ is the same as when this limit is reached by causing x to decrease through positive values.

To find what this limit is, resume the equation

$$\left(1+x\right)^{\frac{1}{x}} = 2 + \frac{1}{2}\left(1-\frac{1}{m}\right) + \frac{1}{2}\frac{1}{3}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)$$
$$+ \frac{1}{2}\frac{1}{3}\frac{1}{4}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\left(1-\frac{3}{m}\right) +$$
$$\frac{1}{2}\frac{1}{3}\frac{1}{4}\frac{1}{5}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\left(1-\frac{3}{m}\right)\left(1-\frac{4}{m}\right),$$

and suppose m to be infinite; then

$$\text{lim.}\left(1+x\right)^{\frac{1}{x}} = 2 + \frac{1}{2} + \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{3}\frac{1}{4} + \frac{1}{2}\frac{1}{3}\frac{1}{4}\frac{1}{5} + \&c.,$$

$$\begin{aligned}
2 &= 2{\cdot}0000000 \ldots \\
\frac{1}{2} &= {\cdot}5000000 \ldots \\
\frac{1}{2.3} &= {\cdot}1666666 \ldots \\
\frac{1}{2.3.4} &= {\cdot}0416666 \ldots \\
\frac{1}{2.3.4.5} &= {\cdot}0083333 \ldots \\
\frac{1}{2.3.4.5.6} &= {\cdot}0013888 \ldots \\
\frac{1}{2.3.4.5.6.7} &= {\cdot}0001984 \ldots \\
\frac{1}{2.3.4.5.6.7.8} &= {\cdot}0000248 \ldots \\
\frac{1}{2.3.4.5.6.7.8.9} &= {\cdot}0000027 \ldots \\
\ldots\ldots &\quad \ldots\ldots
\end{aligned}$$

Sum $= 2{\cdot}7182818 \ldots$, a number that is incommensurable with unity.

It is the base of the Napierian system of logarithms, and will be denoted by e:

Therefore

$$(1+x)^{\frac{1}{x}}_{x=0}{}^{*} = 2{\cdot}7182818 = e.$$

The symbol l will be hereafter used to designate Napierian logarithms, and L will denote other logarithms.

10. Taking the Napierian logarithm of $(1+x)^{\frac{1}{x}}$, we have

$$l\,(1+x)^{\frac{1}{x}} = \frac{1}{x}\,l\,(1+x) = \frac{l\,(1+x)}{x};$$

and, passing to the limit by making $x = 0$, we have

$$\lim.\ \frac{l(1+x)}{x} = \lim.\ l\,(1+x)^{\frac{1}{x}} = le = 1.$$

In any other system of logarithms, we should have

$$L(1+x)^{\frac{1}{x}} = \frac{1}{x}\ L(1+x) = \frac{L(1+x)}{x};$$

and at the limit

$$\lim.\ \frac{L\,(1+x)}{x} = \lim.\ L\,(1+x)^{\frac{1}{x}} = Le = \frac{1}{la},$$

a being the base of the system characterized by L, and observing that, since the logarithms of the same number in two systems are as the moduli of those systems, we have

$$Le : le :: M : 1, \text{ or } Le : 1 :: M : 1$$
$$La : la :: M : 1, \text{ or } \quad 1 : la :: M : 1;$$

and therefore $Le = M = \dfrac{1}{la}$, the modulus of the system of which a is the base.

11. Since $\lim.\ (1+x)^{\frac{1}{x}} = e$ when x is decreased without

* The notation $\left(1+x\right)^{\frac{1}{x}}_{x=0}$ indicates that the value of the expression corresponding to $x = 0$ is taken.

limit, we can from this deduce the limit of $(1+az)^{\frac{1}{z}}$ in which a is any constant quantity.

Thus,

$$(1+az)^{\frac{1}{z}} = \left((1+az)^{\frac{1}{az}}\right)^a.$$

Now, as z diminishes without limit, az will also diminish without limit; and therefore

$$\text{lim. } (1+az)^{\frac{1}{az}} = e;$$

$$\therefore \qquad \text{lim. } (1+az)^{\frac{1}{z}} = e^a.$$

12. In any system of logarithms,

$$L(1+z)^{\frac{1}{z}} = \frac{1}{z}\, L(1+z);$$

$$\therefore \qquad \text{lim. } \frac{L(1+z)}{z} = \text{lim. } L(1+z)^{\frac{1}{z}} = Le;$$

and, if the logarithm be taken in the Napierian System,

$$\text{lim. } \frac{l(1+z)}{z} = le = 1.$$

13. Resuming the equation

$$L(1+z)^{\frac{1}{z}} = \frac{L(1+z)}{z},$$

and making $1+z=a^v$, whence (taking logarithms in the system of which a is the base), $v = L(1+z)$ and $z = a^v - 1$; therefore

$$L(1+z)^{\frac{1}{z}} = \frac{v}{a^v - 1};$$

or, by taking the reciprocals,

$$\frac{1}{L(1+z)^{\frac{1}{z}}} = \frac{a^v - 1}{v}.$$

Now, as z diminishes without limit, so also will v, and they will reach the limit zero together; therefore

$$\lim. \frac{1}{L(1+z)^{\frac{1}{z}}} = \lim. \frac{a^v - 1}{v};$$

$$\text{but } \lim. \frac{1}{L(1+z)^{\frac{1}{z}}} = \frac{1}{Le}; \therefore$$

$$\lim. \frac{a^v - 1}{v} = \frac{1}{Le} = la \text{ when } v = 0.$$

Suppose $a = e^m$, whence $m = la$;
and therefore

$$\lim. \frac{e^{mv} - 1}{v} = le^m = m.$$

14. To define some of the terms, and explain the meaning of some of the symbols, employed in the calculus, let us take the explicit function of a single variable

$$y = f(x),$$

and give to x an increment denoted by Δx; y will receive the corresponding increment

$$\Delta y = \Delta f(x) = f(x + \Delta x) - f(x),$$

and therefore

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

When $\Delta x = 0$, the ratio $\frac{\Delta y}{\Delta x}$ takes the form $\frac{0}{0}$; yet it has in fact a determinate value, which is generally some other function of x, and expresses, as will be seen presently, the tangent of the angle that a straight line, tangent to the curve of which $y = f(x)$ is the equation, makes with the axis of the variable x. This limiting value of the ratio of the increment of the variable to the corresponding increment of the function is called the differential co-efficient, or derivative of the function, and is represented by the notations

$$y', f'(x), \frac{dy}{dx}, \lim. \frac{\Delta y}{\Delta x} = \lim. \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

It is to be observed that the characters Δ, d, are not factors, but symbols of abbreviation; the former signifying increment, difference, or change in value, without reference to amount; while the latter is restricted to particular increments called *differentials*, having such values, that the ratio of the differential of the variable to that of the function is equal to the differential co-efficient or derivative of the function. The differentials are usually regarded as infinitely small.

15. Before the ratio $\frac{\Delta y}{\Delta x}$ reaches its limit $f'(x)$, it must differ from it by some quantity which is a function of Δx, and which vanishes when $\Delta x = 0$. We may therefore write,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x) + \gamma,$$

and, by clearing of fractions,

$$\Delta y = f(x + \Delta x) - f(x) = f'(x)\Delta x + \gamma\Delta x.$$

From this it is seen, that, as the ratio $\frac{\Delta y}{\Delta x}$ approaches its limit $f'(x)$, γ must approach zero; and when Δx, and consequently Δy, becomes infinitely small, γ must also become infinitely small, and should therefore be neglected in comparison with the finite quantity $f'(x)$. We shall then have

$$\frac{\Delta y}{\Delta x} = f'(x) = \frac{dy}{dx}, \quad \Delta y = f'(x)\,\Delta x = dy.$$

It is therefore true, that, when the differential of a function is infinitely small, it is sensibly equal to the increment of the function.

These considerations are of importance, and are made by many authors the basis of the definition of the differential of a function; viz., "The differential of a function of a sin-

gle variable is the first term in the development of the difference between the primitive state of the function and the new state which arises from giving to the variable an increment called the differential of the variable; the development being arranged according to the ascending powers of the increment."

16. The definition of the differential of a function follows from that of the differential co-efficient. It is the product of the differential of the independent variable by the differential co-efficient of the function.

The object of the differential calculus is to explain the modes of passing from all known functions to their differential co-efficients, and the application of the properties of such co-efficients and the corresponding differentials to the investigation of various questions in pure and applied mathematics.

The operation of deriving from functions their differential co-efficients is called differentiation.

SECTION II.

DIFFERENTIAL CO-EFFICIENTS OF EXPLICIT FUNCTIONS OF A SINGLE VARIABLE.

17. It will be convenient, before proceeding to establish rules for finding the differential co-efficients of the different kinds of explicit functions of a single variable, to investigate certain principles which are applicable to all forms.

Constants connected with functions by the signs plus or minus disappear in the process of differentiation.

The increments of the function and the variable will be characterized by the symbol Δ when they are written in the first members of equations; but the labor of making the transformations sometimes required in the second members will be lessened by representing the increment of the variable by the single letter h, which will, of course, be equal to Δx.

Let $y = f(x) \pm c$, and give to the variable in this equation the increment h; then

$$y + \Delta y = f(x+h) \pm c;$$

therefore

$$\Delta y = f(x+h) - f(x),$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}.$$

Passing to the limit by making $\Delta x = h = 0$,

$$\lim. \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x),$$

and

$$dy = f'(x)\, dx.$$

This differential co-efficient is manifestly the same as that which would have been found had there been no constant united to $f(x)$ by the sign plus or minus. As, from their very nature, constants admit of no change of value, c has the same value in the new that it had in the primitive state of the function, and must therefore disappear in the subtraction by which the increment of the function is obtained.

Ex. 1. $\quad y = a + x, \quad \frac{dy}{dx} = 1, \quad dy = dx.$

Ex. 2. $\quad y = a - x, \quad \frac{dy}{dx} = -1, \quad dy = -dx.$

18. If a function of a variable be multiplied or divided by a constant, the differential co-efficient will also be multiplied or divided by the constant.

Let $\quad y = cf(x),$

then $\quad y + \Delta y = cf(x + h),$

$$\Delta y = cf(x + h) - cf(x) = c\big(f(x + h) - f(x)\big),$$

$$\frac{\Delta y}{\Delta x} = \frac{c\big(f(x + h) - f(x)\big)}{h}.$$

Passing to the limit,

$$\lim. \frac{\Delta y}{\Delta x} = cf'(x) = \frac{dy}{dx},$$

and

$$dy = cf'(x)\,dx.$$

Again: let

$$y = \frac{1}{c}f(x),$$

then

$$\Delta y = \frac{1}{c}\big(f(x + h) - f(x)\big),$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{c}\,\frac{\big(f(x + h) - f(x)\big)}{h};$$

$\therefore$

$$\lim. \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{1}{c}f'(x),$$

and $$dy = \frac{1}{c} f'(x)\, dx.$$

Ex. 1. $y = ax + b,\ \frac{dy}{dx} = a,\ dy = adx.$

Ex. 2. $y = \frac{x}{a} - b,\ \frac{dy}{dx} = \frac{1}{a},\ dy = \frac{1}{a}\, dx.$

19. The differential co-efficient of the algebraic sum of several functions of the same variable is the algebraic sum of the differential co-efficients of the separate functions.

Let $$y = f(x) \pm \varphi(x) \pm \psi(x) \pm \ldots,$$

then $$y + \Delta y = f(x+h) \pm \varphi(x+h) \pm \psi(x+h) \pm \ldots$$

$$\Delta y = f(x+h) - f(x) \pm \Big(\varphi(x+h) - \varphi(x)\Big)$$
$$\pm \Big(\psi(x+h) - \psi(x)\Big) \pm \ldots$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h} \pm \frac{\varphi(x+h) - \varphi(x)}{h}$$
$$\pm \frac{\psi(x+h) - \psi(x)}{h} \pm \ldots :$$

whence, passing to the limits, and using the previous notation,

$$\lim. \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x) \pm \varphi'(x) \pm \psi'(x) \pm \ldots,$$

and $$dy = f'(x)\, dx \pm \varphi'(x)\, dx \pm \psi'(x)\, dx \pm \ldots$$
$$= \Big(f'(x) \pm \varphi'(x) \pm \psi'(x) \pm \ldots\Big) dx.$$

Ex. 1. $$y = ax - bx + c$$
$$\frac{dy}{dx} = a - b,\ dy = (a - b)\, dx.$$

Ex. 2. $$y = af(x) \pm b\sqrt{-1}\, \varphi(x)$$

$$\frac{dy}{dx} = af'(x) \pm b\sqrt{-1}\, \varphi'(x)$$

$$dy = \Big(af'(x) \pm b\sqrt{-1}\, \varphi'(x)\Big)\, dx.$$

20. The differential co-efficient of the product of two functions of the same variable is the sum of the products obtained by multiplying each function by the differential co-efficient of the other.

Let $$y = f(x) \times \varphi(x),$$
then $$y + \Delta y = f(x+h) \times \varphi(x+h)$$
$$\Delta y = f(x+h) \times \varphi(x+h) - f(x) \times \varphi(x)$$
$$= \Big(f(x+h) - f(x)\Big)\varphi(x+h) + \Big(\varphi(x+h) - \varphi(x)\Big)f(x);$$
$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}\varphi(x+h) + \frac{\varphi(x+h) - \varphi(x)}{h}f(x).$$

Passing to the limit, by making $\Delta x = h = 0$, we see that

$$\lim. \frac{f(x+h) - f(x)}{h} = f'(x), \ \lim. \varphi(x+h) = \varphi(x)$$
$$\lim. \frac{\varphi(x+h) - \varphi(x)}{h} = \varphi'(x): \text{ hence}$$
$$\lim. \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x) \times \varphi(x) + \varphi'(x) \times f(x).$$

Dividing this equation by $y = f(x) \times \varphi(x)$, member by member, we have

$$\frac{\frac{dy}{dx}}{y} = \frac{f'(x)}{f(x)} + \frac{\varphi'(x)}{\varphi(x)}.$$

21. The rule just demonstrated for finding the differential co-efficient of the product of two functions of the same variable may be extended to the product of any number of functions.

Let $$y = f(x) \times \varphi(x) \times \psi(x),$$
and make $$F(x) = \varphi(x) \times \psi(x);$$
then $$y = f(x) \times F(x),$$
and $$\frac{dy}{dx} = f'(x) \times F(x) + F'(x) \times f(x);$$
but $$u' = F'(x) = \varphi'(x) \times \psi(x) + \psi'(x) \times \varphi(x).$$

Substituting, in the value of $\frac{dy}{dx}$ for $F(x)$ and $F'(x)$, their values, we have

$$\frac{dy}{dx} = \varphi(x)\,\psi(x)\,f'(x) + f(x)\,\psi(x)\,\varphi'(x) + f(x)\,\varphi(x)\,\psi'(x).$$

This process has been carried far enough to discover the law, that the differential co-efficient of the product of any number of functions of the same variable is the algebraic sum of the products found by multiplying the differential co-efficient of each function by the product of all the other functions.

Ex. 1. $y = (a + bx)\,(c - ax)\,mx$

$$\frac{dy}{dx} = b(c - ax)\,mx - a\,(a + bx)\,mx + m\,(a + bx)\,(c - ax)$$

$$= m\Big(ac + (2bc - 2a^2 - 3abx)x\Big)$$

$$dy = m\Big(ac + (2bc - 2a^2 - 3abx)x\Big)dx.$$

22. The differential co-efficient of the quotient of two functions of the same variable is equal to the divisor multiplied by the differential co-efficient of the dividend minus the dividend multiplied by the differential co-efficient of the divisor, the result divided by the square of the divisor.

Let $y = \frac{f(x)}{\varphi(x)}$, then $y + \Delta y = \frac{f(x+h)}{\varphi(x+h)}$

$$\Delta y = \frac{f(x+h)}{\varphi(x+h)} - \frac{f(x)}{\varphi(x)}$$

$$= \frac{f(x+h)\,\varphi(x) - \varphi(x+h)\,f(x).}{\varphi(x+h)\,\varphi(x).}$$

$$= \frac{\Big(f(x+h) - f(x)\Big)\varphi(x) - \Big(\varphi(x+h) - \varphi(x)\Big)\,f(x)}{\varphi(x+h)\,\varphi(x)};$$

therefore

$$\frac{\Delta y}{\Delta x} = \frac{\frac{f(x+n) - f(x)}{h}\varphi(x) - \frac{\varphi(x+h) - \varphi(x)}{h}f(x)}{\varphi(x+h)\,\varphi(x)}.$$

Passing to the limit, by making $\Delta x = h = 0$,

$$\lim. \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{f'(x)\,\varphi(x) - \varphi'(x) f(x)}{\left(\varphi(x)\right)^2}.$$

This result may also be obtained thus:

$$y = \frac{f(x)}{\varphi(x)}, \quad \therefore \quad f(x) = y\,\varphi(x);$$

therefore, by Art. 20,

$$f'(x) = y'\,\varphi(x) + \varphi'(x)\,y;$$

therefore

$$y' = \frac{dy}{dx} = \frac{f'(x)}{\varphi(x)} - \frac{f(x)}{\left(\varphi(x)\right)^2}\varphi'(x)$$

$$= \frac{f'(x)\,\varphi(x) - f(x)\,\varphi'(x)}{\left(\varphi(x)\right)^2}.$$

Ex. 1.

$$y = \frac{a + bx}{b + ax}$$

$$\frac{dy}{dx} = \frac{(b + ax)\,b - (a + bx)a}{(b + ax)^2} = \frac{b^2 - a^2}{(b + ax)^2}$$

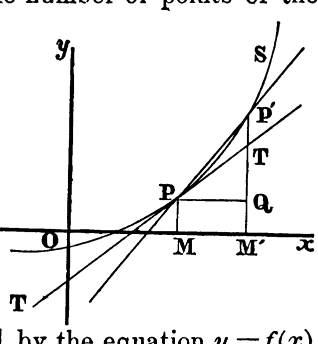
$$dy = \frac{b^2 - a^2}{(b + ax)^2}\,dx.$$

23. The rules which have been thus far demonstrated in this section are independent of the form of the functions characterized by the symbols f, φ, ψ, &c.; and it has been assumed that the differential co-efficients of these component functions of the compound functions considered can be obtained in all cases. Before showing that this assumption is correct, by the actual differentiation of all known forms of simple functions, it is proper to make a few observations on the symbol $\frac{dy}{dx}$, used to denote the differential co-efficient of the function y of the variable x.

In the doctrine of limits, $\frac{dy}{dx}$ represents the limit of the ratio $\frac{\Delta y}{\Delta x}$; and it must be borne in mind, that, at this limit, Δx, and consequently Δy, become zero. There would be, therefore, an inconsistency in viewing dx and dy as the representatives of the terms of a ratio that have vanished, until it be proved that the ratio itself does not also vanish. If the ratio remains, although its terms disappear, then dx and dy may be taken as indeterminates, having for their ratio the final ratio of the vanishing quantities. This is the view to be taken of the differentials dx and dy, according to definition, Art. 14, and which justifies us in regarding these differentials as the terms of a fraction in $\frac{dy}{dx}$.

24. Analytical geometry furnishes instructive illustrations of the meaning of differential co-efficients, as was intimated in Art. 14, and suggests many useful applications that can be made of the doctrine of limits.

Whatever may be the nature of the function $y = f(x)$, every value of x that will give a real value for y will be the abscissa of a point of a curve of which y is the corresponding ordinate; and, if the assumed value of x gives several real values for y, x will be the abscissa of a like number of points of the curve, having for their ordinates the several values of y. The curve is, therefore, the geometrical representative of the relation between x and y in the equation $y = f(x)$.



In the figure, suppose $SP'P$ to be the curve represented by the equation $y = f(x)$,

and let PM be a value of y corresponding to the assumed value OM for x; then give to x the increment $MM' = h$, y will receive the increment $P'Q = \Delta y$, and we have

$$P'Q = \Delta y = f(x+h) - f(x) = P'M' - PM:$$

$$\therefore \qquad \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h} = \frac{P'Q}{PQ}.$$

$\frac{P'Q}{PQ}$ expresses the trigonometrical tangent of the angle $P'PQ$, which is the tangent of the angle that the secant line or chord PP' makes with the axis of the variable x. Now, it is evident, that, as $h = MM'$ diminishes, the point P' moves along the curve towards P, and the secant line approaches coincidence with the tangent line TT'; and finally, when h vanishes, the coincidence of the points, and of the secant with the tangent line is complete. The tangent line to the curve at the point P is then the limiting position of all secant lines which have P for one of the points in which they cut the curve, and $\frac{dy}{dx}$ is the limiting value of the tangents of the angles that such secant lines make with the axis of x.

25. The fraction $\frac{\Delta y}{\Delta x}$ always represents the ratio of the assumed change in the value of the variable to the corresponding change in the value of the function. These changes, when small, are properly called increments; and it is evident that their ratio is the measure of the rate of the increase of the function to that of the variable: but it will be seen, that, for functions in general, this rate of increase will vary both with the initial value of the variable and the value of its increment Δx. If, therefore, the value of the increment were left arbitrary, the value of the fraction $\frac{\Delta y}{\Delta x}$ would be equally so. But the conception of the limiting value of the

ratio removes all uncertainty, and suggests to the mind what the rate of change in the value of the function is, in the immediate vicinity of its value for any assumed value of the variable.

In the case of the curve, in the last article, the limit $\frac{dy}{dx}$ does not depend upon the increment Δx, nor upon the form of the curve at finite distances from the point whose co-ordinates are (x, y), but depends only upon its shape and direction within insensible distances from that point.

26. Let us apply these remarks to the equation $y = \sqrt{2px}$, which is the equation of a parabola referred to its axis, and the tangent line through the vertex as the co-ordinate axes. Giving to x an increment,

$$y + \Delta y = \sqrt{2p(x+h)}$$

$$\Delta y = \sqrt{2p}\left(\sqrt{x+h} - \sqrt{x}\right)$$

$$= \sqrt{2p}\left(\frac{x+h-x}{\sqrt{x+h}+\sqrt{x}}\right) = \frac{h\sqrt{2p}}{\sqrt{x+h}+\sqrt{x}}:$$

$$\therefore \qquad \frac{\Delta y}{\Delta x} = \frac{\sqrt{2p}}{\sqrt{x+h}+\sqrt{x}}:$$

therefore lim. $\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{\sqrt{2p}}{2\sqrt{x}} = \frac{p}{\sqrt{2px}} = \frac{p}{y}.$

From analytical geometry, we know that $\frac{p}{y}$ is the natural tangent of the angle that a tangent line to the parabola, at the point whose ordinate is y, makes with the axis of the curve.

27. The differential co-efficient of a function, which is a power of the variable denoted by any constant exponent, is the exponent multiplied by the variable with its original exponent, less one.

Let $$y = x^n,$$
then $$y + \Delta y = (x+h)^n$$
$$\Delta y = (x+h)^n - x^n = x^n \left\{ \left(1 + \frac{h}{x}\right)^n - 1 \right\}:$$
$$\therefore \quad \frac{\Delta y}{\Delta x} = x^{n-1} \frac{x}{h} \left\{ \left(1 + \frac{h}{x}\right)^n - 1 \right\}.$$

Make $\dfrac{x}{h} = \dfrac{1}{t} \therefore t = \dfrac{h}{x}$, and $\left(1 + \dfrac{h}{x}\right)^n - 1 = (1+t)^n - 1$.

Put also $(1+t)^n - 1 = z \therefore (1+t)^n = 1 + z$.

Making these substitutions in the expression for $\dfrac{\Delta y}{\Delta x}$, it becomes

$$\frac{\Delta y}{\Delta x} = x^{n-1} \frac{z}{t}.$$

Both t and z diminish with h, and reach the limit zero simultaneously with it. Taking the Napierian logarithms of both members of the equation $(1+t)^n = 1 + z$, we have

$$n.l\,(1+t) = l(1+z)$$
$$\therefore \quad n = \frac{l(1+z)}{l(1+t)}.$$

But, Art. 10, $\dfrac{l(1+z)}{z}$ and $\dfrac{l(1+t)}{t}$ both have unity for their limit; hence

$$\lim. \frac{\dfrac{l(1+t)}{t}}{\dfrac{l(1+z)}{z}} = \lim. \frac{l(1+t)}{l(1+z)} \frac{z}{t} = \lim. \frac{1}{n} \frac{z}{t} = 1.$$

But, since n is a constant,

$$\lim. \frac{1}{n} \frac{z}{t} = \frac{1}{n} \lim. \frac{z}{t};$$

$$\therefore \quad \frac{1}{n} \lim. \frac{z}{t} = 1 \therefore \lim. \frac{z}{t} = n:$$

therefore $\lim.\frac{\Delta y}{\Delta x} = \lim. x^{n-1}\frac{z}{t} = \frac{dy}{dx} = nx^{n-1}.$

28. The rule of the last article may also be demonstrated as follows: —

Let, as before, $y = x^n$,

then $y + \Delta y = (x+h)^n$

$$\frac{\Delta y}{\Delta x} = \frac{(x+h)^n - x^n}{h} = x^n \frac{\left(\frac{x+h}{x}\right)^n - 1}{h}.$$

Now, whether n be a whole number or a fraction, positive or negative, it may be represented by $\frac{p-q}{s}$, in which $p, q,$ and s are positive whole numbers.

Make $\frac{x+h}{x} = z \therefore h = x(z-1)$,

and $\frac{\Delta y}{\Delta x} = x^{n-1}\frac{z^n - 1}{z-1}$;

therefore $\lim.\frac{\Delta y}{\Delta x} = \lim. x^{n-1}\frac{z^n - 1}{z-1}.$

As h converges towards its limit zero, z converges towards the limit unity, and h and z reach their respective limits simultaneously: we have then to find the limit of $\frac{z^n - 1}{z-1} = \frac{z^{\frac{p-q}{s}} - 1}{z-1}.$

Make $u = z^{\frac{1}{s}}$; whence $z^{\frac{p-q}{s}} = u^{p-q}$, $z = u^s$, and the limit of u is unity also. Making these substitutions, we have

$$\frac{z^{\frac{p-q}{s}} - 1}{z-1} = \frac{u^{p-q} - 1}{u^s - 1} = \frac{u^p - u^q}{u^q(u^s - 1)} = \frac{u^p - 1 - (u^q - 1)}{u^q(u^s - 1)}.$$

Dividing both numerator and denominator of this last fraction by $u - 1$, it becomes

$$\frac{u^{p-1} + u^{p-2} + \ldots + 1 - (u^{q-1} + u^{q-2} + \ldots + 1)}{u^q(u^{s-1} + u^{s-2} + \ldots + 1)};$$

and, at the limit where $u = 1$, this reduces to $\frac{p-q}{s}$;

hence $\lim. \frac{\Delta y}{\Delta x} = \lim. x^{n-1} \frac{z^n - 1}{z - 1} = \frac{dy}{dx} = \frac{p-q}{s} x^{n-1} = nx^{n-1}$.

29. The rule proved in Arts. 27 and 28 is general; but, when the exponent n is a positive whole number, the demonstration below is more simple than either of those given.

Let $y = x_1 \times x_2 \times \ldots x_n$, in which x_1, x_2, &c., are all functions of the variable x; then, Art. 21,

$$y' = \frac{dy}{dx} =$$

$$x_1 \times x_2 \times \ldots x_{n-1} \times x'_n + x_1 \times x_2 \times \ldots x_{n-2} \times x_n \times x'_{n-1} + \text{\&c.}$$
to n terms.

Now suppose $x_1 = x = x_2 = \ldots = x_n$, then $x'_n = \frac{dx}{dx} = 1 = x'_{n-1} = \ldots = x'$; and each term in the value of y' becomes x^{n-1}: hence $y' = \frac{dy}{dx} = nx^{n-1}$.

Under this supposition in reference to n, we may develop $(x+h)^n$ by the Binomial Formula, and thus get an expression for the ratio $\frac{\Delta y}{\Delta x}$, of which the limit can be readily obtained.

Thus $$y = x^n,\ \Delta y = (x+h)^n - x^n$$

$$= nx^{n-1} h + n \frac{n-1}{2} x^{n-2} h^2 + \text{\&c.}$$

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + n \frac{n-1}{2} x^{n-2} h + \text{\&c.},$$

in which all the terms in the second member after the first term contain h as a factor;

hence $$\lim. \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = nx^{n-1}.$$

Ex. 1. $$y = a + bx^3, \quad \frac{dy}{dx} = 3bx^2.$$

Ex. 2. $$y = \frac{a}{x^2} = ax^{-2}, \quad \frac{dy}{dx} = -\frac{2a}{x^3}.$$

Ex. 3. $$y = \frac{x^2}{x^2 + a^2} = x^2(x^2 + a^2)^{-1}.$$

$$\frac{dy}{dx} = 2x(x^2 + a^2)^{-1} - 2x^3(x^2 + a^2)^{-2}$$

$$= \frac{2x}{x^2 + a^2} - \frac{2x^3}{(x^2 + a^2)^2} = \frac{2a^2 x}{(x^2 + a^2)^2}.$$

30. The differential co-efficient of a function of the form $y = a^x$, a being a constant, is the function multiplied by the Napierian logarithm of the constant.

Let $$y = a^x, \text{ then}$$

$$y + \Delta y = a^{x+h} = a^x a^h$$

$$\Delta y = a^x a^h - a^x = a^x (a^h - 1),$$

and $$\frac{\Delta y}{\Delta x} = a^x \frac{a^h - 1}{h}.$$

Passing to limit,

$$\text{lim.} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \text{lim.}\, a^x \frac{a^h - 1}{h} = a^x \,\text{lim.} \frac{a^h - 1}{h}.$$

But, by Art. 13,

$$\text{lim.} \frac{a^h - 1}{h} = la:$$

therefore $$\text{lim.} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = a^x\, la.$$

If $$y = a^{bx}, \text{ then } y = (a^b)^x,$$

and $$\frac{dy}{dx} = a^{bx}\, la^b.$$

Ex. 1. $$y = e^{(b+cx^2)}, \frac{dy}{dx} = e^{(b+cx^2)} \frac{d(b + cx^2)^*}{dx} = 2cxe^{(b+cx^2)}.$$

* The letter d thus written before an expression indicates differentiation. Thus, if $u = a + bx^2$, then $\frac{d(a + bx^2)}{dx}$ is equivalent to $\frac{du}{dx}$.

Ex. 2. $y = e^{x^n}, \dfrac{dy}{dx} = e^{x^n} \dfrac{d.x^n}{dx} = nx^{n-1}e^{x^n}.$

Ex. 3. $y = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$

$$\frac{dy}{dx} = \sqrt{-1}.\, e^{x\sqrt{-1}} - \sqrt{-1}.\, e^{-x\sqrt{-1}}$$

$$= \sqrt{-1}\left(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}\right).$$

Ex. 4. $y = e^{e^x}, \dfrac{dy}{dx} = e^{e^x} e^x.$

31. The differential co-efficient of a function which is the logarithm of the variable taken in any system is the modulus of the system, divided by the variable.

Let $$y = Lx;$$

then $y + \Delta y = L(x+h)$, $\Delta y = L(x+h) - Lx = L\dfrac{x+h}{x}$.

Whence $$\frac{\Delta y}{\Delta x} = \frac{L\dfrac{x+h}{x}}{h}.$$

Make $h = xz$: therefore

$$\frac{\Delta y}{\Delta x} = \frac{L(1+z)}{xz} = \frac{1}{x}\,\frac{L(1+z)}{z}.$$

But h and z reach the limit zero simultaneously; and, by Art. 10, the limit of $\dfrac{L(1+z)}{z}$, for $z = 0$, is equal to $Le = \dfrac{1}{la}$ $= M$, the modulus of the system: therefore

$$\lim.\ \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{M}{x}.$$

Hence, if $$y = lx, \frac{dy}{dx} = \frac{1}{x}.$$

Ex. 1. $$y = lx^n = nlx, \frac{dy}{dx} = n\frac{1}{x}.$$

Ex. 2. $$y = x^2lx, \qquad \frac{dy}{dx} = x + 2xlx.$$

32. The differential co-efficient of the sine of an arc is equal to the cosine of the arc.

Let $y = \sin. x$, then

$$y + \Delta y = \sin. (x + h)$$

$$\Delta y = \sin. (x + h) - \sin. x$$

$$= 2 \cos. \left(x + \frac{h}{2}\right) \sin. \frac{h}{2}. \text{ (Eq. 16, Plane Trig.)}$$

Therefore
$$\frac{\Delta y}{\Delta x} = \frac{\sin. \frac{h}{2}}{\frac{h}{2}} \cos. \left(x + \frac{h}{2}\right).$$

But, when h is diminished indefinitely, the limit of

$$\frac{\sin. \frac{h}{2}}{\frac{h}{2}} = 1 \text{ (Art. 7), and lim. } \cos. \left(x + \frac{h}{2}\right) = \cos. x:$$

therefore
$$\text{lim. } \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \cos. x.$$

33. The differential co-efficient of the cosine of an arc is equal to minus the sine of the arc.

Let $y = \cos. x$, then

$$y + \Delta y = \cos. (x + h)$$

$$\Delta y = \cos. (x + h) - \cos. x$$

$$= -2 \sin. \frac{h}{2} \sin. \left(x + \frac{h}{2}\right)$$

$$\frac{\Delta y}{\Delta x} = -\frac{\sin. \frac{h}{2}}{\frac{h}{2}} \sin. \left(x + \frac{h}{2}\right).$$

At the limit $\dfrac{\sin. \frac{h}{2}}{\frac{h}{2}} = 1, \ \sin. \left(x + \frac{h}{2}\right) = \sin. x:$

therefore

$$\lim. \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = -\sin. x.$$

34. The differential co-efficient of the tangent of an arc is equal to 1 divided by the square of the cosine of the arc.

Let $y = \tan. x$, then

$$y + \Delta y = \tan. (x + h)$$

$$\Delta y = \tan. (x + h) - \tan. x$$

$$= \frac{\sin. (x + h)}{\cos. (x + h)} - \frac{\sin. x}{\cos. x}$$

$$= \frac{\sin. (x + h) \cos. x - \cos. (x + h) \sin. x}{\cos. (x + h) \cos. x}$$

$$= \frac{\sin. (x + h - x)}{\cos. (x + h) \cos. x} \text{ (Eq. 8 Plane Trig.)}$$

$$= \frac{\sin. h}{\cos. (x + h) \cos. x}:$$

therefore

$$\frac{\Delta y}{\Delta x} = \frac{\sin. h}{h} \frac{1}{\cos. (x + h) \cos. x}$$

at the limit $\frac{\sin. h}{h} = 1$, $\frac{1}{\cos. (x + h) \cos. x} = \frac{1}{\cos.^2 x}$;

hence $$\lim. \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{1}{\cos.^2 x} = \sec.^2 x.$$

35. The differential co-efficient of the cotangent of an arc is equal to minus 1 divided by the square of the sine of the arc.

Let $y = \cot. x$, then

$$y + \Delta y = \cot. (x + h).$$

Proceeding with this as in the case of the tangent, we should find

$$\frac{dy}{dx} = -\frac{1}{\sin.^2 x} = -\text{cosec.}^2 x.$$

36. The differential co-efficient of the secant of an arc is equal to the sine of the arc divided by the square of its cosine.

Let $y = \sec. x$, then

$$y + \Delta y = \sec. (x + h)$$

$$\Delta y = \sec. (x + h) - \sec. x$$

$$= \frac{1}{\cos. (x + h)} - \frac{1}{\cos. x} = \frac{\cos. x - \cos. (x + h)}{\cos. x \cos. (x + h)}$$

$$= \frac{2 \sin. \frac{h}{2} \sin. \left(x + \frac{h}{2}\right)}{\cos. x \cos. (x + h)} \quad \text{(Eq. 18 Plane Trig.).}$$

Therefore

$$\frac{\Delta y}{\Delta x} = \frac{\sin. \frac{h}{2}}{\frac{h}{2}} \frac{\sin. \left(x + \frac{h}{2}\right)}{\cos. x \cos. (x + h)}.$$

Passing to the limit,

$$\lim. \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{\sin. x}{\cos.^2 x} = \tan. x \sec. x.$$

37. The differential co-efficient of the cosecant of an arc is equal to minus the cosine of the arc divided by the square of its sine.

Let $y = \text{cosec.}\, x$, then $y + \Delta y = \text{cosec.}\, (x + h)$, and $\Delta y = \text{cosec.}\, (x + h) - \text{cosec.}\, x$, and so on; the process being the same as in the case of the secant. We should thus find

$$\frac{dy}{dx} = -\frac{\cos. x}{\sin.^2 x} = -\text{cosec.}\, x \cot. x.$$

38. The differential co-efficient of the versed-sine of an arc is equal to the sine of the arc.

Let $y = \text{vers.}\, x = 1 - \cos. x$, then we have

$$\frac{dy}{dx} = \frac{d. \text{vers.}\, x}{dx} = \frac{d. (1 - \cos. x)}{dx} = -\frac{d. \cos. x}{dx} = \sin. x.$$

39. The circular functions whose differential co-efficients have been thus far found are called direct circular functions. Since the tangent, cotangent, secant, and cosecant may all be expressed under a fractional form in terms of the sine and cosine, their differential co-efficients could have been found by the rule of Art. 22. Thus: —

1st, $$y = \tan. x = \frac{\sin. x}{\cos. x}$$

$$\frac{dy}{dx} = \frac{\cos.^2 x + \sin.^2 x}{\cos.^2 x} = \frac{1}{\cos.^2 x}.$$

2d, $$y = \cot. x = \frac{\cos. x}{\sin. x}$$

$$\frac{dy}{dx} = -\frac{\sin.^2 x + \cos.^2 x}{\sin.^2 x} = -\frac{1}{\sin.^2 x}.$$

3d, $$y = \sec. x = \frac{1}{\cos. x}$$

$$\frac{dy}{dx} = \frac{\sin. x}{\cos.^2 x} = \tan. x \sec. x.$$

4th, $$y = \operatorname{cosec}. x = \frac{1}{\sin. x}$$

$$\frac{dy}{dx} = -\frac{\cos. x}{\sin.^2 x} = \cot. x \operatorname{cosec}. x.$$

The other forms frequently given to the differential co-efficients of the direct circular functions will be readily recognized by the student familiar with the elementary principles of trigonometry.

SECTION III.

DIFFERENTIAL CO-EFFICIENTS OF INVERSE FUNCTIONS, FUNCTIONS OF FUNCTIONS, AND COMPLEX FUNCTIONS OF A SINGLE VARIABLE.

40. THE inverse circular functions are those in which the sine, cosine, tangent, &c., are taken as the independent variables, the arcs being the functions. They are written $y = \sin.^{-1} x$, $y = \cos.^{-1} x$, $y = \tan.^{-1} x$, &c.; and are read y equal to the arc of which x is the sine, cosine, tangent, &c. These functions are sometimes written $y = \text{arc} \sin. x$, $y = \text{arc} \tan. x$, &c., and also $y = \text{arc} (\sin. = x)$, $y = \text{arc} (\tan. = x)$, &c.; but the first notation, being the shortest, and that generally adopted, will be uniformly used in what follows.

41. If we have $y = \varphi(x)$, then the differential co-efficient of x, regarded as a function of y, is the reciprocal of the differential co-efficient of y regarded as a function of x.

That is, if $y = \varphi(x)$, then x must be some function of y, such as $x = \psi(y)$; whence $\dfrac{dy}{dx} = \varphi'(x)$, $\dfrac{dx}{dy} = \psi'(y)$: and, according to the principle enunciated, we should have

$$\frac{dx}{dy} = \psi'(y) = \frac{1}{\varphi'(x)}.$$

As an example, take the equation

$$y = \varphi(x) = x^2 + 2x - 3;$$

from which we get

$$\frac{dy}{dx} = 2(x+1).$$

Solving the equation with respect to x, we have

$$x = -1 \pm \sqrt{y+4};$$

whence $\qquad \dfrac{dx}{dy} = \pm \dfrac{1}{2\sqrt{y+4}}$; but $\pm\sqrt{y+4} = x+1$:

therefore $\qquad \dfrac{dx}{dy} = \dfrac{1}{2(x+1)},$

which accords with the theorem; and we will now prove that what holds in this particular case is true for all cases.

Let $y = \varphi(x)$ . . . (1) be the given function; and since, from the nature of equations, x must also be some function of y, suppose $x = \psi(y)$ . . . (2) to be that function.

If, in Eq. 1, x receives the increment Δx, y will receive a corresponding increment Δy: therefore

$$y + \Delta y = \varphi(x + \Delta x) \quad . \; . \; . \quad (3).$$

Now, Eqs. 1 and 2 are but different forms of the expression of a certain relation between the variables x and y; and whatever values of y in Eq. 1 result from an assigned value to x, if one of these values of y be assigned to y in Eq. 2, then, among the different resulting values of x, one at least must be the value assigned to x in Eq. 1.

It is therefore proper to assume that x and y have the same values in Eq. 2 that they have respectively in Eq. 1. Change, then, in Eq. 2, y into $y + \Delta y$, and x into $x + \Delta x$, these symbols having the same values that they have in Eq. 3: hence

$$x + \Delta x = \psi(y + \Delta y) \quad . \; . \; . \quad (4).$$

From Eqs. 1 and 3, we have

$$\frac{\Delta y}{\Delta x} = \frac{\varphi(x + \Delta x) - \varphi(x)}{\Delta x} \quad . \; . \; . \quad (5);$$

from Eqs. 2 and 4,

$$\frac{\Delta x}{\Delta y} = \frac{\psi(y + \Delta y) - \psi(y)}{\Delta y} \quad . \; . \; . \; (6):$$

multiplying Eqs. 5 and 6, member by member, then

$$\frac{\Delta y}{\Delta x} \times \frac{\Delta x}{\Delta y} = \frac{\varphi(x + \Delta x) - \varphi(x)}{\Delta x} \times \frac{\psi(y + \Delta y) - \psi(y)}{\Delta y} \; (7).$$

By the preceding remarks, $\frac{\Delta y}{\Delta x} \times \frac{\Delta x}{\Delta y} = 1$; and, in the second member of Eq. 7, the first factor at the limit becomes $\frac{dy}{dx} = \varphi'(x)$; and the second factor, $\frac{dx}{dy} = \psi'(y)$: hence

$$\frac{dy}{dx}\frac{dx}{dy} = \varphi'(x)\psi'(y) = 1.$$

Whence $\frac{dx}{dy} = \frac{1}{\varphi'(x)} = \psi'(y) = \frac{1}{\frac{dy}{dx}}$.

42. If we have

$$y = \psi(z) \quad . \; . \; . \; (1)$$

$$z = \varphi(x) \quad . \; . \; . \; (2):$$

then y is a function of x; for, by substituting in the first of these equations the expression for z in the second, y becomes an explicit function of x. Suppose this to be denoted by

$$y = f(x) \quad . \; . \; . \; (3):$$

Now, if x, in Eq. 2, receive the increment Δx, z will take the increment Δz; and, in consequence of this increment of z, y in Eq. 1 will become $y + \Delta y$: hence we should have, from Eqs. 1 and 2,

$$y + \Delta y = \psi(z + \Delta z) \quad . \; . \; . \; (4)$$

$$z + \Delta z = \varphi(x + \Delta x) \quad . \; . \; . \; (5);$$

also, from Eq. 3,

$$y + \Delta y = f(x + \Delta x) \quad . \; . \; . \; (6).$$

From the mode of dependence of the variables, we may assume that the symbols x, y, z, Δx, Δy, Δz, have respectively the same values in all of the preceding equations in which they occur. Subtracting Eq. 3 from Eq. 6, member from member, and dividing both members of the result by Δx, we have

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \ldots (7);$$

similarly, from Eqs. 1 and 4,

$$\frac{\Delta y}{\Delta z}=\frac{\psi(z+\Delta z)-\psi(z)}{\Delta z} \quad \ldots (8);$$

and, from Eqs. 2 and 5,

$$\frac{\Delta z}{\Delta x}=\frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x} \quad \ldots (9).$$

Multiplying Eqs. 8 and 9, member by member, we have, because the symbols are supposed to have the same values throughout,

$$\frac{\Delta y}{\Delta z}\frac{\Delta z}{\Delta x}=\frac{\Delta y}{\Delta x}=\frac{\psi(z+\Delta z)-\psi(z)}{\Delta z}\,\frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x} \quad . . (10);$$

equating the second members of Eqs. 7 and 10,

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\psi(z+\Delta z)-\psi(z)}{\Delta z}\times\frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x}.$$

Whence, passing to the limit,

$$f'(x)=\psi'(z)\,\varphi'(x),$$

or

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}.$$

Ex. 1. $y=z^2+3z-5 \quad . . . (1), \quad z=2x-3 \quad . . . (2)$

$$\frac{dy}{dz}=2z+3, \quad \frac{dz}{dx}=2, \quad \frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=4z+6$$
$$=8x-6.$$

By placing for z, in Eq. 1, its value from Eq. 2, we find

$$y = 4x^2 - 6x - 5, \text{ whence } \frac{dy}{dx} = 8x - 6;$$

the same result as was found by the first process.

43. Differential co-efficients of the inverse circular functions.

1st, Differential co-efficient of $y = \sin.^{-1} x$.

Since $y = \sin.^{-1} x$, $x = \sin. y$; and therefore, by Art. 32,

$$\frac{dx}{dy} = \cos. y = \pm \sqrt{1 - x^2};$$

and therefore, by Art. 41,

$$\frac{dy}{dx} = \frac{1}{\cos. y} = \pm \frac{1}{\sqrt{1 - x^2}}.$$

2d, Differential co-efficient of $y = \cos.^{-1} x$.

Here $y = \cos.^{-1} x$ gives $x = \cos. y$: therefore, Art. 33,

$$\frac{dx}{dy} = -\sin. y = \mp \sqrt{1 - x^2};$$

and therefore, by Art. 41,

$$\frac{dy}{dx} = -\frac{1}{\sin. y} = \mp \frac{1}{\sqrt{1 - x^2}}.$$

It would be superfluous to point out the necessity for the signs $\pm$, $\mp$, before the differential co-efficients in this and the preceding case.

3d, Differential co-efficient of $y = \tan.^{-1} x$.

From $y = \tan.^{-1} x$, we have $x = \tan. y$: therefore

$$\frac{dx}{dy} = \frac{1}{\cos.^2 y} = \sec.^2 y = 1 + \tan.^2 y \quad \text{(Art. 34)};$$

and $$\frac{dy}{dx} = \cos.^2 y = \frac{1}{1 + \tan.^2 y} \quad \text{(Art. 41)}.$$

Whence $$\frac{dy}{dx} = \frac{1}{1 + x^2}.$$

4th, Differential co-efficient of $y = \cot.^{-1} x$.

From $y = \cot.^{-1} x$, we have $x = \cot. y$: therefore

$$\frac{dx}{dy} = -\frac{1}{\sin.^2 y} = -\operatorname{cosec.}^2 y = -(1 + \cot.^2 y) \text{ (Art. 35).}$$

$$\frac{dy}{dx} = -\sin.^2 y = -\frac{1}{1+x^2} \text{ (Art. 41).}$$

5th, Differential co-efficient of $y = \sec.^{-1} x$.

From $y = \sec.^{-1} x$, we have $x = \sec. y$: therefore

$$\frac{dx}{dy} = \frac{\sin. y}{\cos.^2 y} = \sec.^2 y \sin. y \text{ (Art. 36);}$$

and $$\frac{dy}{dx} = \frac{\cos.^2 y}{\sin. y} = \frac{1}{\sec.^2 y \sin. y} \text{ (Art. 41).}$$

But $$\sec. y = \frac{1}{\cos. y}, \text{ hence } \cos. y = \frac{1}{\sec. y} = \frac{1}{x}; \text{ and}$$

$$1 - \sin.^2 y = \frac{1}{x^2}, \ \sin. y = \pm \frac{\sqrt{x^2 - 1}}{x}: \text{ therefore}$$

$$\frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2 - 1}}.$$

6th, Differential co-efficient of $y = \operatorname{cosec.}^{-1} x$.

We shall merely indicate the steps.

$$x = \operatorname{cosec.} y, \ \frac{dx}{dy} = -\frac{\cos. y}{\sin.^2 y} = -\operatorname{cosec.} y \cot. y \text{ (Art. 37):}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\operatorname{cosec.} y \cot. y}, \quad \sin. y = \frac{1}{\operatorname{cosec.} y} = \frac{1}{x};$$

$$\therefore \quad \cot. y = \pm\sqrt{x^2 - 1}; \ \therefore \frac{dy}{dx} = \mp \frac{1}{x\sqrt{x^2 - 1}}.$$

7th, Differential co-efficient of $y = \operatorname{vers.}^{-1} x$.

Taking x for the function, we have

$$x = \operatorname{vers.} y = 1 - \cos. y;$$

therefore $\frac{dx}{dy} = \sin. y$ (Art. 38), and $\frac{dy}{dx} = \frac{1}{\sin. y}$ (Art. 41):

but $\frac{1}{\sin. y} = \pm \frac{1}{\sqrt{1 - \cos.^2 y}} = \pm \frac{1}{\sqrt{1 - (1 - \text{vers.} y)^2}}$

$$= \pm \frac{1}{\sqrt{1 - (1 - x)^2}} = \pm \frac{1}{\sqrt{2x - x^2}};$$

$$\therefore \qquad \frac{dy}{dx} = \pm \frac{1}{\sqrt{2x - x^2}}.$$

44. The principle demonstrated in Art. 41 has greatly simplified the investigation of the formulas expressing the differential co-efficients of the inverse trigonometrical functions. They may, however, be determined directly, without the aid of this principle.

We will illustrate the manner in which this may be done by a single example:—

Let $$y = \sin.^{-1} x, \text{ then}$$

$$y + \Delta y = \sin.^{-1} (x + h);$$

and $$\Delta y = \sin.^{-1} (x + h) - \sin.^{-1} x.$$

The second member of this last equation is the difference of two arcs whose respective sines are $x + h$ and x; and this difference is, by trigonometry (Plane Trig., Eq. 8), equal to an arc having for its sine the sine of the first arc multiplied by the cosine of the second, minus the cosine of the first multiplied by the sine of the second. Expressing the cosines of these arcs in terms of their sines, we have

$$\Delta y = \sin.^{-1} (x + h) - \sin.^{-1} x$$

$$= \sin.^{-1} \left((x+h)\sqrt{1 - x^2} - x\sqrt{[1 - (x + h)^2]}\right):$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{\sin.^{-1} \left((x+h)\sqrt{1 - x^2} - x\sqrt{[1 - (x + h)^2]}\right)}{h}.$$

Make $z = (x+h)\sqrt{1-x^2} - x\sqrt{[1-(x+h)^2]}$:

therefore $$\frac{\Delta y}{\Delta x} = \frac{\sin.^{-1} z}{h} = \frac{\sin.^{-1} z}{z}\,\frac{z}{h}.$$

Now z and h diminish together, and become zero simultaneously. At the limit, $\frac{\sin.^{-1} z}{z} = 1$. To find what $\frac{z}{h}$ becomes at the same time, multiply and divide the expression for z by $(x+h)\sqrt{1-x^2} + x\sqrt{\left(1-(x+h)^2\right)}$; then

$$\frac{z}{h} = \frac{(x+h)^2(1-x^2) - x^2\left(1-(x+h)^2\right)}{h\left((x+h)\sqrt{1-x^2} + x\sqrt{[1-(x+h)^2]}\right)}$$

$$= \frac{2x+h}{(x+h)\sqrt{1-x^2} + x\sqrt{\left(1-(x+h)^2\right)}}.$$

Pass to the limit by making $h = 0$, and we have

$$\text{lim. } \frac{z}{h} = \frac{x}{x\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}:$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

45. Differential co-efficients of functions of the form $y = t^s$ in which t and s are both functions of the same variable x.

Taking the Napierian logarithms of both members of the equation $y = t^s$, we have $ly = slt$. By Art. 42, the differential co-efficient with respect to x of ly is

$\frac{d.ly}{dx} = \frac{1}{y}\frac{dy}{dx}$; and, by Arts. 20, 42, that of slt is

$$lt\,\frac{d.s}{ds}\,\frac{ds}{dx} + s\,\frac{d.lt}{dt}\,\frac{dt}{dx} = lt\,\frac{ds}{dx} + s\,\frac{1}{t}\,\frac{dt}{dx}.$$

Now, since the equation $ly = slt$ is true for all values of x,

the differential co-efficients of its two members must also be equal: therefore

$$\frac{1}{y}\frac{dy}{dx} = lt\frac{ds}{dx} + \frac{s}{t}\frac{dt}{dx};$$

whence $$\frac{dy}{dx} = y\left(lt\frac{ds}{dx} + \frac{s}{t}\frac{dt}{dx}\right)$$

$$= t^s\, lt\, \frac{ds}{dx} + st^{s-1}\frac{dt}{dx} = t^s\left(lt\frac{ds}{dx} + \frac{s}{t}\frac{dt}{dx}\right).$$

46. From an examination of the particular cases treated in Arts. 19, 20, 45, we deduce this general rule for finding the differential co-efficient of any compound function: Differentiate each component function in succession, treating the others as constant, and take the algebraic sum of the results.

Rules have now been given for the differentiation of all known forms of algebraic, logarithmic, exponential, and circular functions of a single variable; and we have seen, that, in general, the differential co-efficients of these functions are themselves functions of the same variable.

47. The following exercises are given that the preceding rules may be impressed, and that the students may become expert in their application, and familiar with the forms of the differential co-efficients of simple and complex functions:—

1. $y = ax^3,\ \frac{dy}{dx} = 3ax^2$ (Art. 28).

2. $y = abx^3 - cx^2$

$$\frac{dy}{dx} = 3abx^2 - 2cx \text{ (Arts. 19, 28).}$$

3. $y = \frac{ax^2}{(b - x^2)^3},$

$$\frac{dy}{dx} = \frac{2ax(b-x^2)^3 + 6ax^3(b-x^2)^2}{(b-x^2)^6} \text{ (Arts. 21, 22).}$$

$$= \frac{2ax(b + 2x^2)}{(b - x^2)^4}.$$

4. $$y=\sqrt{x+\sqrt[3]{x+c}}=\left(x+(x+c)^{\frac{1}{3}}\right)^{\frac{1}{2}}.$$

Put $z=x+(x+c)^{\frac{1}{3}}$; then $y=z^{\frac{1}{2}}$,

and $$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx} \quad \text{(Art. 42).}$$

But $$\frac{dy}{dz}=\frac{1}{2z^{\frac{1}{2}}}=\frac{1}{2\left(x+(x+c)^{\frac{1}{3}}\right)^{\frac{1}{2}}},$$

and $$\frac{dz}{dx}=1+\frac{1}{3}(x+c)^{-\frac{2}{3}}=\frac{3(x+c)^{\frac{2}{3}}+1}{3(x+c)^{\frac{2}{3}}}:$$

therefore $$\frac{dy}{dx}=\frac{1}{2\left(x+(x+c)^{\frac{1}{3}}\right)^{\frac{1}{2}}}\,\frac{3(x+c)^{\frac{2}{3}}+1}{3(x+c)^{\frac{2}{3}}}$$

$$=\frac{3\sqrt[3]{(x+c)^2}+1}{6\sqrt{x+\sqrt[3]{x+c}}\times\sqrt[3]{(x+c)^2}}.$$

5. $y=l(x+\sqrt{1+x^2})$. Make $z=x+\sqrt{1+x^2}$;

then $$y=lz, \text{ and } \frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}.$$

But $$\frac{dy}{dz}=\frac{1}{z}=\frac{1}{x+\sqrt{1+x^2}} \quad \text{(Art. 31).}$$

and $$\frac{dz}{dx}=1+\frac{x}{\sqrt{1+x^2}}=\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}:$$

therefore $$\frac{dy}{dx}=\frac{1}{x+\sqrt{1+x^2}}\,\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}.$$

The utility of substituting a single symbol to represent a complicated expression before differentiation is exemplified in this and the preceding examples. Oftentimes the labor, both mental and mechanical, of the mathematician is greatly abridged by the adoption of suitable artifices.

6. $y = l\,\dfrac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}+x}$. Multiply both numerator and denominator by the numerator, then

$$y = l\left(1+2x^2-2x\sqrt{1+x^2}\right).$$

Put $\quad z = 1+2x^2-2x\sqrt{1+x^2}$. Whence $y = lz$,

and
$$\frac{dy}{dx} = \frac{dy}{dz}\,\frac{dz}{dx};$$

but
$$\frac{dy}{dz} = \frac{1}{z} = \frac{1}{1+2x^2-2x\sqrt{1+x^2}},$$

and
$$\frac{dz}{dx} = 4x - 2\sqrt{1+x^2} - \frac{2x^2}{\sqrt{1+x^2}} =$$

$$-\frac{2\left(1+2x^2-2x\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}:\ \therefore$$

$$\frac{dy}{dx} = \frac{1}{1+2x^2-2x\sqrt{1+x^2}} \times -\frac{2\left(1+2x^2-2x\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}$$

$$= -\frac{2}{\sqrt{1+x^2}}.$$

7. $$y = \tan.^{-1}\frac{3a^2x-x^3}{a(a^2-3x^2)}.$$

Make
$$\frac{3a^2x-x^3}{a(a^2-3x^2)} = z;\ \text{then } y = \tan.^{-1}z,$$

and
$$\frac{dy}{dx} = \frac{dy}{dz}\,\frac{dz}{dx},\quad \frac{dy}{dz} = \frac{1}{1+z^2}$$

$$\frac{1}{1+z^2} = \frac{1}{1+\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right)^2} = \frac{a^2(a^2-3x^2)^2}{(a^2+x^2)^3}$$

$$\frac{dz}{dx} = \frac{3(a^2-x^2)(a^2-3x^2)+6x(3xa^2-x^3)}{a(a^2-3x^2)^2}$$

$$= \frac{3(a^4+2a^2x^2+x^4)}{a(a^2-3x^2)^2} = \frac{3(a^2+x^2)^2}{a(a^2-3x^2)^2}:$$

therefore $$\frac{dy}{dx}=\frac{dy}{dz}\,\frac{dz}{dx}=\frac{a^2(a^2-3x^2)^2}{(a^2+x^2)^3}\times\frac{3(a^2+x^2)^2}{a(a^2-3x^2)^2};$$

$$\therefore \quad \frac{dy}{dx}=\frac{3a}{a^2+x^2}.$$

This is as it should be; for

$$\tan.^{-1}\frac{3a^2x-x^3}{a(a^2-3x^2)}=3\tan.^{-1}\frac{x}{a}*:$$

therefore $y=3\tan.^{-1}\frac{x}{a}$. Make $\frac{x}{a}=z$, then

$$y=3\tan.^{-1}z, \text{ and } \frac{dy}{dx}=\frac{dy}{dz}\,\frac{dz}{dx};$$

but $$\frac{dy}{dz}=\frac{3}{1+z^2}=\frac{3}{1+\frac{x^2}{a^2}}=\frac{3a^2}{a^2+x^2} \quad \text{(Art. 43)},$$

and $$\frac{dz}{dx}=\frac{1}{a}: \quad \therefore \frac{dy}{dx}=\frac{3a}{a^2+x^2}.$$

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* To prove this, take Eqs. 28 and 33 Plane Trig., and in 28 make $b=2a$; then

$$\tan.\,3a=\frac{\tan.\,a+\tan.\,2a}{1-\tan.\,a\tan.2a}.$$

Substituting in the second member of this the value of $\tan.\,2a$ taken from Eq. 33, and reducing, we find

$$\tan.\,3a=\frac{3\tan.\,a-\tan.^3a}{1-3\tan.^2a}.$$

Dividing both numerator and denominator of the fraction $\frac{3a^2x-x^3}{a^3-3ax^2}$ by a^3, it becomes $\dfrac{3\frac{x}{a}-\frac{x^3}{a^3}}{1-3\frac{x^2}{a^2}}$; and, comparing this with the formula for the tangent of $3a$, we conclude that it is the expression for the tangent of three times the arc of which $\frac{x}{a}$ is the tangent: therefore

$$3\tan.^{-1}\frac{x}{a}=\tan.^{-1}\frac{3\frac{x}{a}-\frac{x^3}{a^3}}{1-3\frac{x^2}{a^2}}=\tan.^{-1}\frac{3a^2x-x^3}{a(a^2-3x^2)},$$

as was assumed.

8. $$y = \frac{e^{ax}(a \sin. x - \cos. x)}{a^2 + 1}$$

$$\frac{dy}{dx} = \frac{1}{a^2 + 1} \frac{d}{dx} e^{ax}(a \sin. x - \cos. x)$$

$$\frac{d}{dx} e^{ax}(a \sin. x - \cos. x) = ae^{ax}(a \sin. x - \cos. x)$$

$$+ e^{ax}(a \cos. x + \sin. x)$$

$$= (a^2 + 1)e^{ax} \sin. x:$$

$\therefore$ $$\frac{dy}{dx} = e^{ax} \sin. x.$$

9. $$y = \frac{1}{\sqrt{a^2 - b^2}} \sin.^{-1} \frac{b + a \sin. x}{a + b \sin. x}.$$

Put $\dfrac{b + a \sin. x}{a + b \sin. x} = z$; then $y = \dfrac{1}{\sqrt{a^2 - b^2}} \sin.^{-1} z$,

and $$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx}.$$

But

$$\frac{dy}{dz} = \frac{1}{\sqrt{a^2 - b^2}} \frac{1}{\sqrt{1 - z^2}} = \frac{1}{\sqrt{a^2 - b^2}} \frac{1}{\sqrt{1 - \left(\frac{b + a \sin. x}{a + b \sin. x}\right)^2}}$$

$$= \frac{a + b \sin. x}{\sqrt{a^2 - b^2} \cos. x},$$

and $$\frac{dz}{dx} = \frac{(a + b \sin. x) a \cos. x - (b + a \sin. x) b \cos. x}{(a + b \sin. x)^2}$$

$$= \frac{(a^2 - b^2) \cos. x}{(a + b \sin. x)^2}:$$

therefore $$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - b^2}} \frac{a + b \sin. x}{\sqrt{a^2 - b^2} \cos. x} \frac{(a^2 - b^2) \cos. x}{(a + b \sin. x)^2}$$

$$= \frac{1}{a + b \sin. x}.$$

10. If $y = \frac{1}{\sqrt{a^2 - b^2}} \cos.^{-1} \frac{b + a \cos. x}{a + b \cos. x}$, we should find, in like manner,

$$\frac{dy}{dx} = \frac{1}{a + b \cos. x}.$$

11. $$y = \tan.^{-1}\left(\frac{e^x \cos. x}{1 + e^x \sin. x}\right).$$

Put $$z = \frac{e^x \cos. x}{1 + e^x \sin. x}; \text{ then } y = \tan.^{-1} z,$$

and $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{1 + z^2}\frac{dz}{dx}$$

$$\frac{dz}{dx} = \frac{(e^x \cos. x - e^x \sin. x)(1 + e^x \sin. x) - e^x \cos. x(e^x \cos. x + e^x \sin. x)}{(1 + e^x \sin. x)^2}$$

$$= \frac{e^x(\cos. x - \sin. x - e^x)}{(1 + e^x \sin. x)^2},$$

and $$\frac{1}{1+z^2} = \frac{1}{1 + \left(\frac{e^x \cos. x}{1 + e^x \sin. x}\right)^2} = \frac{(1 + e^x \sin. x)^2}{(1 + e^x \sin. x)^2 + (e^x \cos. x)^2}$$

$$= \frac{(1 + e^x \sin. x)^2}{1 + 2e^x \sin. x + e^{2x}}:$$

therefore $$\frac{dy}{dx} = \frac{e^x(\cos. x - \sin. x - e^x)}{1 + 2e^x \sin. x + e^{2x}}.$$

12. $$y = \frac{1 + x}{1 + x^2}, \qquad \frac{dy}{dx} = \frac{1 - 2x - x^2}{(1 + x^2)^2}.$$

13. $$y = xlx, \qquad \frac{dy}{dx} = 1 + lx.$$

14. $$y = l \cot. x, \qquad \frac{dy}{dx} = -\frac{2}{\sin. 2x}.$$

15. $$y = \frac{x}{\sqrt{(a^2 - x^2)}}, \qquad \frac{dy}{dx} = \frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

16. $y = e^x(1 - x^3), \quad \dfrac{dy}{dx} = e^x(1 - 3x^2 - x^3).$

17. $y = \left(\dfrac{x}{a}\right)^{ax}, \quad \dfrac{dy}{dx} = a\left(\dfrac{x}{a}\right)^{ax}\left(1 + l\dfrac{x}{a}\right).$

18. $y = \dfrac{x^n}{(1+x)^n}, \quad \dfrac{dy}{dx} = \dfrac{nx^{x-1}}{(1+x)^{n+1}}.$

19. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \dfrac{dy}{dx} = \dfrac{4}{(e^x + e^{-x})^2}.$

20. $y = l(e^x + e^{-x}), \quad \dfrac{dy}{dx} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}.$

21. $y = (a+x)^m (b+x)^n,$

$$\frac{dy}{dx} = (a+x)^{m-1}(b+x)^{n-1}\Big\{m(b+x) + n(a+x)\Big\}.$$

22. $y = \dfrac{\tan.^3 x}{3} - \tan. x + x, \quad \dfrac{dy}{dx} = \tan.^4 x.$

23. $y = \dfrac{1}{x + \sqrt{1-x^2}}, \quad \dfrac{dy}{dx} = \dfrac{x - \sqrt{1-x^2}}{\sqrt{1-x^2}\,(1 + 2x\sqrt{1-x^2})}.$

24. $y = (a^2 + x^2)\tan.^{-1}\dfrac{x}{a}, \quad \dfrac{dy}{dx} = 2x\tan.^{-1}\dfrac{x}{a} + a.$

25. $y = l\Big\{l(a + bx^n)\Big\}, \quad \dfrac{dy}{dx} = \dfrac{nbx^{n-1}}{(a+bx^n)\,l(a+bx^n)}.$

26. $y = l\tan.\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right), \quad \dfrac{dy}{dx} = \dfrac{1}{\cos. x}.$

27. $y = \dfrac{\sqrt{a+x}}{\sqrt{a} + \sqrt{x}}, \quad \dfrac{dy}{dx} = \dfrac{\sqrt{ax} - a}{2\sqrt{ax + x^2}\,(\sqrt{a} + \sqrt{x})^2}.$

28. $y = \sqrt{\dfrac{1+x}{1-x}}, \quad \dfrac{dy}{dx} = \dfrac{1}{(1-x)\sqrt{1-x^2}}.$

29. $y = \dfrac{x}{e^x - 1}, \quad \dfrac{dy}{dx} = \dfrac{e^x(1-x) - 1}{(e^x - 1)^2}.$

30. $y = l\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}, \quad \dfrac{dy}{dx} = -\dfrac{1}{x\sqrt{1-x^2}}.$

31. $y = \left(\dfrac{x}{1+\sqrt{1-x^2}}\right)^n, \quad \dfrac{dy}{dx} = \dfrac{ny}{x\sqrt{1-x^2}}.$

32. $y = a^{\frac{1}{\sqrt{a^2-x^2}}}, \quad \dfrac{dy}{dx} = \dfrac{xy}{(a^2-x^2)^{\frac{3}{2}}} la.$

33. $y = e^x\left(\dfrac{x+1}{x-1}\right)^{\frac{1}{2}}, \quad \dfrac{dy}{dx} = e^x \dfrac{x^2-2}{(x+1)^{\frac{1}{2}}(x-1)^{\frac{3}{2}}}.$

34. $y = \dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}, \quad \dfrac{dy}{dx} = -\dfrac{2}{x^3}\left(1+\dfrac{1}{\sqrt{1-x^4}}\right).$

35. $y = \sin.^{-1} mx, \quad \dfrac{dy}{dx} = \dfrac{m}{(1-m^2x^2)^{\frac{1}{2}}}.$

36. $y = x\sin.^{-1} x, \quad \dfrac{dy}{dx} = \sin.^{-1} x + \dfrac{x}{\sqrt{1-x^2}}.$

37. $y = e^{\tan.^{-1} x}, \quad \dfrac{dy}{dx} = e^{\tan.^{-1} x} \dfrac{1}{1+x^2}.$

38. $y = x^{\sin.^{-1} x}, \quad \dfrac{dy}{dx} = x^{\sin.^{-1} x}\left(\dfrac{xlx + (1-x^2)^{\frac{1}{2}}\sin.^{-1} x}{x(1-x^2)^{\frac{1}{2}}}\right).$

39. $y = \dfrac{x - \sin.^{-1} x}{\sin.^3 x},$

$$\frac{dy}{dx} = \frac{\sin. x\left(1 - \dfrac{1}{\sqrt{1-x^2}}\right) - 3(x - \sin.^{-1} x)\cos. x.}{\sin.^4 x}.$$

40. $y = l\dfrac{a + b\tan.\frac{x}{2}}{a - b\tan.\frac{x}{2}}, \quad \dfrac{dy}{dx} = \dfrac{ab}{a^2\cos.^2\frac{x}{2} - b^2\sin.^2\frac{x}{2}}.$

41. $y = x^{\frac{1}{x}}, \quad \dfrac{dy}{dx} = \dfrac{x^{\frac{1}{x}}(1-lx)}{x^2}.$

42. $y = e^{e^x}$, $\qquad \dfrac{dy}{dx} = e^{e^x} e^x.$

43. $y = x^{x^x}$, $\qquad \dfrac{dy}{dx} = x^{x^x} x^x \left(\dfrac{1}{x} + lx + (lx)^2 \right).$

44. $y = x^{e^x}$, $\qquad \dfrac{dy}{dx} = x^{e^x} e^x \dfrac{1 + xlx}{x}.$

45. $y = \sin.^{-1} \dfrac{x+1}{\sqrt{2}}$, $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - 2x - x^2}}.$

46. $y = \tan.^{-1} \dfrac{x}{\sqrt{1-x^2}}$, $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}.$

47. $y = \tan. \sqrt{1-x}$,

$$\frac{dy}{dx} = -\frac{1}{(\cos. \sqrt{1-x})^2} \frac{1}{2\sqrt{1-x}} = -\frac{(\sec. \sqrt{1-x})^2}{2\sqrt{1-x}}.$$

48. $y = \tan.^{-1} (n \tan. x)$, $\qquad \dfrac{dy}{dx} = \dfrac{n}{\cos.^2 x + n^2 \sin.^2 x}.$

49. $y = (a + x) \tan.^{-1} \sqrt{\dfrac{x}{a}} - \sqrt{ax}$, $\qquad \dfrac{dy}{dx} = \tan.^{-1} \sqrt{\dfrac{x}{a}}.$

50. $y = \tan.^{-1} \dfrac{2x}{1+x^2}$, $\qquad \dfrac{dy}{dx} = \dfrac{2(1-x^2)}{1 + 6x^2 + x^4}.$

51. $y = \tan.^{-1} \dfrac{2x}{1-x^2}$, $\qquad \dfrac{dy}{dx} = \dfrac{2}{1+x^2}.$

52. $y = \sin.^{-1} \sqrt{\sin. x}$, $\qquad \dfrac{dy}{dx} = \dfrac{1}{2} \sqrt{1 + \operatorname{cosec.} x}.$

53. $y = \sin.^{-1} \dfrac{ax}{b + cx^2}$,

$$\frac{dy}{dx} = \frac{a(b - cx^2)}{(b + cx^2)\sqrt{b^2 + (2bc - a^2)x^2 + c^2 x^4}}.$$

54. $y = \sqrt{1-x^2} \sin.^{-1} x - x$, $\qquad \dfrac{dy}{dx} = -\dfrac{x \sin.^{-1} x}{\sqrt{1-x^2}}.$

55. $y = \frac{x \sin.^{-1} x}{\sqrt{1 - x^2}} + l\sqrt{1 - x^2}, \quad \frac{dy}{dx} = \frac{\sin.^{-1} x}{(1 - x^2)^{\frac{3}{2}}}.$

56. $y = \sin.^{-1} \frac{x \tan. b}{\sqrt{a^2 - x^2}}, \quad \frac{dy}{dx} = \frac{a^2 \tan. b}{a^2 - x^2} \frac{1}{(a^2 - x^2 \sec.^2 b)^{\frac{1}{2}}}.$

57. $y = \sin.^{-1} \sqrt{\frac{a^2 - x^2}{b^2 - x^2}}, \quad \frac{dy}{dx} = -\frac{x\sqrt{b^2 - a^2}}{(b^2 - x^2)\sqrt{a^2 - x^2}}.$

58. $y = \tan.^{-1} \frac{\sqrt{1 - \cos. x}}{\sqrt{1 + \cos. x}}, \quad \frac{dy}{dx} = \frac{1}{2}.$

59. $y = \tan.^{-1} \left(\frac{(a^2 - b^2)^{\frac{1}{2}} \sin. x}{b + a \cos. x} \right), \quad \frac{dy}{dx} = \frac{(a^2 - b^2)^{\frac{1}{2}}}{a + b \cos. x}.$

60. $y = \sec.^{-1} \frac{1}{2x^2 - 1}, \quad \frac{dy}{dx} = -\frac{2}{\sqrt{1 - x^2}}.$

61. $y = \tan.^{-1} \frac{\sqrt{1 + x^2} - 1}{x}, \quad \frac{dy}{dx} = \frac{1}{2(1 + x^2)}.$

62. $y = l \frac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2} + 2 \tan.^{-1} \frac{x\sqrt{2}}{1 - x^2}, \quad \frac{dy}{dx} = \frac{4\sqrt{2}}{1 + x^4}.$

63. $y = \frac{1}{6} l \frac{(t + 1)^2}{t^2 - t + 1} - \frac{1}{\sqrt{3}} \tan.^{-1} \frac{2t - 1}{\sqrt{3}}$, in which

$$t = \frac{(1 + 3x + 3x^2)^{\frac{1}{3}}}{x}, \quad \frac{dy}{dx} = \frac{1}{xt(1 + x)}.$$

64. From the equation

$$\sin. x + \sin. 2x + \ldots + \sin. nx = \frac{\sin. \frac{n + 1}{2} x \sin. \frac{n}{2} x}{\sin. \frac{1}{2} x},$$

prove, by differentiating both members, that

$\cos. x + 2 \cos. 2x + 3 \cos. 3x + \ldots + n \cos. nx$ is equal to

$$\frac{\frac{n + 1}{2} \sin. \frac{1}{2} x \sin. \frac{2n + 1}{2} x - \frac{1}{2} \left(\sin. \frac{n + 1}{2} x \right)^2}{\sin.^2 \frac{1}{2} x}.$$

65. Admitting* that

$$\sin. x \sin\left(\frac{\pi}{m}+x\right)\sin.\left(\frac{2\pi}{m}+x\right)\ldots\sin.\left(\frac{m-1}{m}\pi+x\right)$$
$$=\frac{\sin. mx}{2^{m-1}},$$

in which m is a positive whole number, prove that

$$\cot. x+\cot.\left(\frac{\pi}{m}+x\right)+\ldots\cot.\left(\frac{m-1}{m}\pi+x\right)=m\cot. mx.$$

---

* As the equations assumed in this and the preceding example are not usually given in treatises on elementary trigonometry, they will be demonstrated in the Key to this work.

SECTION IV.

SUCCESSIVE DIFFERENTIAL CO-EFFICIENTS.

48. THE differential co-efficient of a function, $f(x)$, of a single variable, being in general another function, $f'(x)$, of the same variable, we may subject this new function to the rules by which $f'(x)$ was derived from $f(x)$, and thus obtain the second derivative, or differential co-efficient, of the original function. The second differential co-efficient will, in turn, give rise to a third, and so on; and we thus arrive at the successive derivatives, or differential co-efficients, of a function.

The notation by which these successive differential co-efficients are indicated will be best explained by an example:—

Let us take $y = x^n$; then

$$\frac{dy}{dx} = y' = nx^{n-1} \quad \ldots\ldots \quad 1^{\text{st}} \text{ diff. co-efficient.}$$

$$\frac{d^2y}{dx^2} = y'' = n(n-1)x^{n-2} \quad \ldots \quad 2^{\text{d}} \text{ diff. co-efficient.}$$

. . . .

$$\frac{d^my}{dx^m} = y^{(m)} \quad \ldots\ldots\ldots \quad m^{\text{th}} \text{ diff. co-efficient.}$$

These are the first, second, . . . m^{th} differential co-efficients of the function $y = f(x)$. It is sometimes convenient to denote these by writing the function itself with as many dashes as there have been differentiations performed: thus $f'(x)$, $f''(x)$, . . . $f^{(m)}(x)$, are the first, second, . . . m^{th} differen-

tial co-efficients of $f(x)$, and have the same signification as $y,' y'', \ldots y^{(m)}$.

In the example just given, it is evident, that, if n be a positive integer, the m^{th} differential co-efficient will be independent of x, that is, a constant, when $m = n$; and that the function will not have a differential co-efficient of a higher order than the n^{th}. In other cases and forms of function, there will be no limit to the number of differentiations that may be performed.

The symbols $\dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}, \ldots \dfrac{d^my}{dx^m}$, are

equivalent to $\dfrac{d\dfrac{dy}{dx}}{dx}, \dfrac{d\dfrac{d^2y}{dx^2}}{dx}, \ldots \dfrac{d\dfrac{d^{m-1}y}{dx^{m-1}}}{dx}$, and

are read second, third, ... m^{th} differential co-efficient of y regarded as a function of x; and are to be viewed as wholes, and not as fractions, having d^2y, d^3y, ... d^my, for their numerator, and dx^2, dx^3, ... dx^m, for their denominators: nor must the indices 2, 3, ... m, be considered as exponents of powers, but as denoting the number of times the function has been differentiated.

49. Successive differential co-efficients of the product of two functions of the same variable. Leibnitz' Theorem.

Take $u = yz$, in which y and z are functions of x; then, by Art. 20, we have

$$\frac{du}{dx} = y\frac{dz}{dx} + z\frac{dy}{dx};$$

and, differentiating both members of this equation with respect to x, we have

$$\frac{d^2u}{dx^2} = y\frac{d^2z}{dx^2} + \frac{dy}{dx}\frac{dz}{dx} + \frac{dy}{dx}\frac{dz}{dx} + z\frac{d^2y}{dx^2}$$

$$= y\frac{d^2z}{dx^2} + 2\frac{dy}{dx}\frac{dz}{dx} + z\frac{d^2y}{dx^2}.$$

In like manner, we should find

$$\frac{d^3u}{dx^3}=y\frac{d^3z}{dx^3}+3\frac{dy}{dx}\frac{d^2z}{dx^2}+3\frac{dz}{dx}\frac{d^2y}{dx^2}+z\frac{d^3y}{dx^3},$$

and

$$\frac{d^4u}{dx^4}=y\frac{d^4z}{dx^4}+4\frac{dy}{dx}\frac{d^3z}{dx^3}+6\frac{d^2y}{dx^2}\frac{d^2z}{dx^2}+4\frac{dz}{dx}\frac{d^3y}{dx^3}+z\frac{d^4y}{dx^4}.$$

This has been carried far enough to enable us to discover inferentially the laws which govern the numerical co-efficients, and the indices of differentiation in the expressions for $\frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}, \frac{d^4u}{dx^4}$. These laws are the same as those for the co-efficients and exponents in the Binomial Formula; for, in respect to differentiation, y may be regarded as $y^{(0)}$, and z as $z^{(0)}$.

To prove these laws to be general, let us assume them to hold when n is the index of differentiation. Then

$$\frac{d^nu}{dx^n}=y\frac{d^nz}{dx^n}+n\frac{dy}{dx}\frac{d^{n-1}z}{dx^{n-1}}+n\frac{n-1}{2}\frac{d^2y}{dx^2}\frac{d^{n-2}z}{dx^{n-2}}+\ldots$$

$$+n\frac{(n-1)(n-2)\ldots(n-r+1)}{2.3.\ldots r}\frac{d^ry}{dx^r}\frac{d^{n-r}z}{dx^{n-r}}$$

$$+n\frac{(n-1)(n-2)\ldots(n-r)}{2.3.\ldots r(r+1)}\frac{d^{r+1}y}{dx^{r+1}}\frac{d^{n-(r+1)}z}{dx^{n-(r+1)}}$$

$$+\ldots+z\frac{d^ny}{dx^n}.$$

Differentiating both members of this equation with respect to x, reducing, and arranging the result, we find

$$\frac{d^{n+1}u}{dx^{n+1}}=y\frac{d^{n+1}z}{dx^{n+1}}+(n+1)\frac{dy}{dx}\frac{d^nz}{dx^n}+\ldots$$

$$+\frac{(n+1)n\ldots(n+1-r)}{2.3.\ldots(r+1)}\frac{d^{r+1}y}{dx^{r+1}}\frac{d^{n-r}z}{dx^{n-r}}+\ldots+z\frac{d^{n+1}y}{dx^{n+1}}.$$

Now, the laws of the co-efficients and indices in this development are the same as those assumed to be true in that from which it was immediately derived; but, by actual operation, we know them to hold when $n = 4$: they therefore hold when $n = 5$; and so on: hence they are universal.

As an example of the above, take $u = e^{ax} y$; then, observing that $\frac{d^n e^{ax}}{dx^n} = a^n e^{ax}$, we find

$$\frac{d^n u}{dx^n} = e^{ax}\left(a^n y + na^{n-1}\frac{dy}{dx} + n\frac{n-1}{2}a^{n-2}\frac{d^2 y}{dx^2} + \ldots + \frac{d^n y}{dx^n}\right).$$

Now, by examining the expression within the parentheses, we discover, that if $\left(a + \frac{d}{dx}\right)^n y$ be developed by the Binomial Theorem, treating the symbol $\frac{d}{dx}$ as a quantity, and $\left(\frac{d}{dx}\right) y$ $\left(\frac{d}{dx}\right)^2 y \ldots \left(\frac{d}{dx}\right)^n y$ be then replaced by $\frac{dy}{dx}, \frac{d^2 y}{dx^2} \ldots \frac{d^n y}{dx^n}$, we get that factor of the development of $\frac{d^n u}{dx^n}$: hence

$$\frac{d^n u}{dx^n} = \frac{d^n (e^{ax} y)}{dx^n} = e^{ax}\left(a + \frac{d}{dx}\right)^n y$$

is a convenient and abridged form of writing the n^{th} differential co-efficient of the function $u = e^{ax} y$.

50. If n be a positive whole number, we may prove that

$$v\frac{d^n u}{dx^n} = \frac{d^n uv}{dx^n} - n\frac{d^{n-1}}{dx^{n-1}}\left(u\frac{dv}{dx}\right) + \frac{n(n-1)}{1.2}\frac{d^{n-2}}{dx^{n-2}}\left(u\frac{d^2 v}{dx^2}\right)$$

$$- \&c. + \&c. \ldots + (-1)^n u\frac{d^n v}{dx^n} \ldots (1).$$

For let $y = uv$, in which both u and v are functions of x; then, differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d.uv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx},$$

$$\ldots z\frac{d^3y}{dx^3},$$

$$\ldots + z\frac{d^4y}{dx^4}.$$

... us to discover ... co-efficients, ... expressions for ... same as those for the ... Formula: for, in ... as $y^{(0)}$, and

... assume them ... Then

$$\ldots + \ldots$$

$$\ldots \frac{\ldots (n-r+1)}{\ldots r}\frac{d^ry}{dx^r}\frac{d^{n-r}z}{dx^{n-r}}$$

$$\ldots \frac{\ldots (n-r)}{\ldots (r+1)}\frac{d^{r+1}y}{dx^{r+1}}\frac{d^{n-(r+1)}z}{dx^{n-(r+1)}}$$

Now, the laws of the co-efficients and indices in this development are the same as those assumed to be true in that from which it was immediately derived; but, by actual operation we know them to hold when $n = 4$; they therefore hold when $n = 5$; and so on: hence they are universal.

As an example of the above, take $u = e^{ax}y$; then, observing that $\frac{d^r e^{ax}}{dx^r} = a^r e^{ax}$, we find

$$\frac{d^n u}{dx^n} = e^{ax}\left(a^n y + n a^{n-1}\frac{dy}{dx} + n\frac{n-1}{1.2}a^{n-2}\frac{d^2 y}{dx^2} + \ldots + \frac{d^n y}{dx^n}\right).$$

Now, by examining the expression within the parentheses, we discover, that if $\left(a + \frac{d}{dx}\right)^n y$ be developed by the Binomial Theorem, treating the symbol $\frac{d}{dx}$ as a quantity, and $\left(\frac{d}{dx}\right) y$, $\left(\frac{d}{dx}\right)^2 y$, ... $\left(\frac{d}{dx}\right)^n y$ be then replaced by $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, ... $\frac{d^n y}{dx^n}$, we get that factor of the development of $\frac{d^n u}{dx^n}$: hence

$$\frac{d^n u}{dx^n} = \frac{d^n (e^{ax} y)}{dx^n} = e^{ax}\left(a + \frac{d}{dx}\right)^n y$$

is a convenient and abridged form of writing the n^{th} differential co-efficient of the function $u = e^{ax}y$.

50. If n be a positive whole number, we may prove that

$$\frac{d^n u}{\ldots} = \frac{d^n \ldots}{\ldots} \ldots \frac{d^{n-1}}{dx^{n-1}}\left(u\frac{dv}{dx}\right) + \frac{n(n-1)}{1.2}\frac{d^{n-2}}{dx^{n-2}}\left(u\frac{d^2 v}{dx^2}\right)$$

$$\ldots + (-1)^n u \frac{d^n v}{dx^n} \ldots (1).$$

... both u and v are functions of ... respect to x, we have

$$\ldots = u\frac{dv}{dx} + v\frac{du}{dx} \ldots$$

whenco $$v\frac{du}{dx}=\frac{d.\,uv}{dx}-u\frac{dv}{dx};$$

and the theorem holds when $n=1$.

Now, differentiate both members of Eq. 1; then

$$v\frac{d^{n+1}u}{dx^{n+1}}+\frac{dv}{dx}\frac{d^nu}{dx^n}=\frac{d^{n+1}uv}{dx^{n+1}}-n\frac{d^n}{dx^n}\left(u\frac{dv}{dx}\right)$$
$$+\frac{n(n-1)}{2}\frac{d^{n-1}}{dx^{n-1}}\left(u\frac{d^2v}{dx^2}\right)$$
$$-\text{\&c.}+\ldots+(-1)^n\frac{d}{dx}\left(u\frac{d^nv}{dx^n}\right)\ldots(2).$$

If the theorem holds for $y=uv$ when the index n has some assigned value, it will also hold for $u\frac{dv}{dx}$ when n has the same value. Changing, in Eq. 1, v into $\frac{dv}{dx}$, we have

$$\frac{dv}{dx}\frac{d^nu}{dx^n}=\frac{d^n}{dx^n}\left(u\frac{dv}{dx}\right)-n\frac{d^{n-1}}{dx^{n-1}}\left(u\frac{d^2v}{dx^2}\right)$$
$$+\frac{n(n-1)}{1.2}\frac{d^{n-2}}{dx^{n-2}}\left(u\frac{d^3v}{dx^3}\right)$$
$$-\text{\&c.}+\ldots+(-1)^n u\frac{d^{n+1}v}{dx^{n+1}}\ldots(3).$$

Subtracting Eq. 3 from 2, member from member, and reducing, we find

$$v\frac{d^{n+1}u}{dx^{n+1}}=\frac{d^{n+1}uv}{dx^{n+1}}-(n+1)\frac{d^n}{dx^n}\left(u\frac{dv}{dx}\right)$$
$$+(n+1)\frac{n}{2}\frac{d^{n-1}}{dx^{n-1}}\left(u\frac{d^2v}{dx^2}\right)-\text{\&}+\ldots$$
$$+(-1)^{n+1}u\frac{d^{n+1}v}{dx^{n+1}}.$$

Hence, if the theorem is true for any assigned value of n, it is true when the index is $n+1$. It is true when $n=1$; it

is therefore true when $n = 2$; and so on; that is, it is universally true.

EXAMPLES.

1. $y = lx, \ \frac{dy}{dx} = \frac{1}{x}, \ \frac{d^2 y}{dx^2} = -\frac{1}{x^2}$

$$\frac{d^3 y}{dx^3} = \frac{1.2}{x^3}, \ \frac{d^4 y}{dx^4} = -\frac{1.2.3}{x^4}$$

$$\frac{d^n y}{dx^n} = (-1)^{n-1} \frac{1.2.3.(n-1)}{x^n}.$$

2. $y = \sin. x, \ \frac{dy}{dx} = \cos. x = \sin. \left(x + \frac{\pi}{2}\right),$

$$\frac{d^2 y}{dx^2} = \frac{d \sin. \left(x + \frac{\pi}{2}\right)}{dx} = \cos. \left(x + \frac{\pi}{2}\right) = \sin. \left(x + \frac{2\pi}{2}\right),$$

$$\frac{d^3 y}{dx^2} = \frac{d \sin. \left(x + \frac{2\pi}{2}\right)}{dx} = \cos. \left(x + \frac{2\pi}{2}\right) = \sin. \left(x + \frac{3\pi}{2}\right),$$

. . . .

$$\frac{d^n y}{dx^n} = \sin. \left(x + \frac{n\pi}{2}\right).$$

3. $y = \cos. x, \ \frac{dy}{dx} = -\sin. x = \cos. \left(x + \frac{\pi}{2}\right),$

. . . .

$$\frac{d^n y}{dx^n} = \cos. \left(x + \frac{n\pi}{2}\right).$$

4. $y = \cos. ax, \ \frac{d^n y}{dx^n} = a^n \cos. \left(ax + \frac{n\pi}{2}\right).$

5. $y = \tan. x + \sec. x,$

$$\frac{dy}{dx} = \frac{1}{\cos.^2 x} + \frac{\sin. x}{\cos.^2 x} = \frac{1 + \sin. x}{\cos.^2 x} = \frac{1}{1 - \sin. x},$$

$$\frac{d^2 y}{dx^2} = \frac{\cos. x}{(1 - \sin. x)^2}$$

6. $y = \sin.^3 x = \frac{3 \sin. x - \sin. 3x}{4}$,

$$\frac{d^n y}{dx^n} = \frac{3}{4} \sin. \left(x + \frac{n\pi}{2}\right) - \frac{3^n}{4} \sin. \left(3x + \frac{n\pi}{2}\right).$$

7. $y = x^2 lx, \quad \frac{d^3 y}{dx^3} = \frac{1.2}{x}.$

8. $y = x^3 lx, \quad \frac{d^4 y}{dx^4} = \frac{1.2.3}{x}.$

9. $y = x^{n-1} lx, \qquad \frac{d^n y}{dx^n} = \frac{1.2.3\ldots(n-1)}{x}.$

10. $y = (x^2 + a^2) \tan.^{-1} \frac{x}{a}, \quad \frac{d^3 y}{dx^3} = \frac{4a^3}{(a^2 + x^2)^2}.$

11. $y = e^{-x} \cos. x, \qquad \frac{d^4 y}{dx^4} = -4e^{-x} \cos. x;$ hence

$$\frac{d^4 y}{dx^4} + 4y = 0.$$

12. $y = \frac{1-x}{1+x}, \qquad \frac{d^n y}{dx^n} = (-1)^n 2 \frac{1.2.3\ldots n}{(1+x)^{n+1}}.$

13. $y = (e^x + e^{-x})^n$,

$$\frac{d^2 y}{dx^2} = n^2 (e^x + e^{-x})^n - 4n(n-1)(e^x + e^{-x})^{n-2}.$$

14. $y^2 = \sec. 2x, \qquad y + \frac{d^2 y}{dx^2} = 3y^5.$

15. $y = \frac{ax+b}{x^2 - c^2}$,

$$\frac{d^n y}{dx^n} = (-1)^n \frac{1.2\ldots n}{2c} \left(\frac{b+ac}{(x-c)^{n+1}} - \frac{b-ac}{(x+c)^{n+1}}\right).$$

16. $y = \frac{1}{x^2 - a^2}$,

$$\frac{d^n y}{dx^n} = (-1)^n \frac{1.2.3\ldots n}{2a} \left((x-a)^{-(n+1)} - (x+a)^{-(n+1)}\right).$$

17. $y = x^n \sin. x$,

$$\frac{d^n y}{dx^n} = 1.2.3\ldots n \left\{ \sin. x + \frac{n}{1} x \sin.\left(x + \frac{\pi}{2}\right) \right.$$

$$+ \frac{n(n-1)}{1.2.1.2} x^2 \sin. x + \frac{2\pi}{2}$$

$$\left. + \frac{n(n-1)(n-2)}{1.2.3.1.2.3} x^3 \sin.\left(x + \frac{3\pi}{2}\right) + \&c. \right\}.$$

18. $\frac{y}{a} = \tan.^{-1}\frac{x}{a}$, $\frac{dy}{dx} = \cos.^2 \frac{y}{a}$,

$$\frac{d^2 y}{dx^2} = \frac{1}{a} \cos.\left(\frac{2y}{a} + \frac{\pi}{2}\right) \cos.^2 \frac{y}{a},$$

$$\frac{d^3 y}{dx^3} = \frac{2}{a^2} \cos.\left(\frac{3y}{a} + 2\frac{\pi}{2}\right) \cos.^3 \frac{y}{a},$$

. . . .

and $$\frac{d^n y}{dx^n} = \frac{1.2.3\ldots(n-1)}{a^{n-1}} \cos.\left(\frac{ny}{a} + (n-1)\frac{\pi}{2}\right) \cos.^n \frac{y}{a}.$$

Because $\tan.^{-1}\frac{x}{a} = \frac{\pi}{2} - \tan.^{-1}\frac{a}{x}$, make $\tan.^{-1}\frac{a}{x} = \theta$; then

$$\tan.^{-1}\frac{x}{a} = \frac{\pi}{2} - \theta = \frac{y}{a},\quad \frac{ny}{a} = \frac{n\pi}{2} - n\theta$$

$$\cos.\left(\frac{ny}{a} + (n-1)\frac{\pi}{2}\right) = \sin.\left(\frac{ny}{a} + \frac{n\pi}{2}\right) = \sin.(n\pi - n\theta)$$

$$= (-1)^{n-1} \sin. n\theta : \text{ also } \cos. \frac{y}{a} = \frac{a}{(a^2 + x^2)^{\frac{1}{2}}},$$

$$\cos.^n \frac{y}{a} = \frac{a^n}{(a^2 + x^2)^{\frac{n}{2}}}.$$

Substituting these values in the expression for $\frac{d^n y}{dx^n}$, we have

$$\frac{d^n y}{dx^n} = a(-1)^{n-1} \frac{1.2.3\ldots(n-1)}{(a^2 + x^2)^{\frac{n}{2}}} \sin. n\theta.$$

19. Prove that

$$\frac{d^n}{dx^n}\left(\frac{1}{a^2+x^2}\right)=\frac{(-1)^n}{a}\,\frac{1.2.3\ldots n\sin.(n+1)\theta}{(a^2+x^2)^{\frac{n+1}{2}}}.$$

Proceed as follows: —

$$\frac{d\tan.^{-1}\frac{x}{a}}{dx}=\frac{a}{a^2+x^2}\;\therefore\;\frac{d^n}{dx^n}\left(\frac{1}{a^2+x^2}\right)=\frac{1}{a}\,\frac{d^{n+1}\tan.^{-1}\frac{x}{a}}{dx^{n+1}};$$

and, from this point, the process is similar to that pursued in the preceding example.

20. Prove that

$$\frac{d^n}{dx^n}\left(\frac{x}{a^2+x^2}\right)=\frac{(-1)^n 1.2.3\ldots n\cos.(n+1)\theta}{(a^2+x^2)^{\frac{n+1}{2}}}.$$

By Art. 49,

$$\frac{d^n}{dx^n}\left(\frac{x}{a^2+x^2}\right)=x\frac{d^n}{dx^n}\left(\frac{1}{a^2+x}\right)+n\,\frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{a^2+x^2}\right);$$

and finding the value of each term in second member, as in last example, we get

$$\frac{d^n}{dx^n}\left(\frac{x}{a^2+x^2}\right)=x\,\frac{(-1)^n\,1.2.3\ldots n\sin.(n+1)\theta}{a(a^2+x^2)^{\frac{n+1}{2}}}$$

$$+\frac{(-1)^{n-1}\,1.2.3\ldots n\sin.n\theta}{a(a^2+x^2)^{\frac{n}{2}}}$$

$$=\frac{(-1)^n 1.2.3\ldots n\cos.(n+1)\theta}{(a^2+x^2)^{\frac{n+1}{2}}}.$$

21. When $y=\sin.(m\sin.^{-1}x)$, prove that

$$(1-x^2)\frac{d^2y}{dx^2}=x\frac{dy}{dx}-m^2y.$$

22. When $y=a\cos.lx+b\sin.lx$, prove that

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0;$$

also that

$$x^2\frac{d^{n+2}y}{dx^{n+2}}+(2n+1)x\frac{d^{n+1}y}{dx^{n+1}}+(n^2+1)\frac{d^ny}{dx^n}=0.$$

SECTION V.

RELATIONS EXISTING BETWEEN REAL FUNCTIONS OF A SINGLE VARIABLE AND THEIR DIFFERENTIAL CO-EFFICIENTS.—TAYLOR'S THEOREM.—MACLAURIN'S THEOREM.

51. If the increment Δx of the variable x in the function $y = F(x)$ produces the increment Δy of y, then the ratio $\frac{\Delta y}{\Delta x}$, having for its limit $\frac{dy}{dx} = F'(x)$, will, before reaching this limit, and when Δx is very small, take the sign of $F'(x)$; that is, it will be positive if the differential co-efficient is positive, and negative if the differential co-efficient is negative. But, when a ratio is positive, its terms have the same sign; and, when negative, they have opposite signs. Hence, if the differential co-efficient of a function be positive, the function will increase or decrease according as the variable increases or decreases; but, if the differential co-efficient be negative, the function will decrease as the variable increases, and the opposite.

52. Suppose that the function $y = F(x)$ is continuous between the limits answering to the assigned values $x = x_0$, $x = x_1$, of the variable, and that the variable passes by insensible degrees from the first to the second of these values; then, by the foregoing article, the function cannot change from an increasing to a decreasing, or from a decreasing to an increasing function, unless the differential co-efficient changes its sign from positive to negative, or from negative to positive. But a function can change its sign only when it passes

through zero or infinity. If continuous, the change of sign will occur in passing through zero; if discontinuous, in passing through infinity.

53. If the function $y=F(x)$ vanishes for $x=x_0$, and is continuous for values of x which are indefinitely near x_0, then

$$F(x_0+\Delta x)=\Delta x F'(x_0)+\Delta x r \ldots \text{ (Art. 15)},$$

where r is a very small quantity when Δx is very small. The sign of the second member will therefore be determined by that of $F'(x)$: hence, if $x=x_0+\Delta x$ differs very little from x_0,

$$F(x_0+\Delta x)=F(x)>0 \text{ if } F'(x)>0,$$
$$F(x_0+\Delta x)=F(x)<0 \text{ if } F'(x)<0.$$

54. Suppose that the two functions $F(x), f(x)$, are real, and that they, as well as their differential co-efficients, are continuous between the limits, answering to the values x_1 and x_1+h of the variable; suppose also, that, between these limits, $f'(x)$ does not undergo a change of sign; that is, for the intermediate values of x, $f(x)$ must constantly be either an increasing or a decreasing function: then the ratio of the differences

$$F(x_1+h)-F(x_1),\ f(x_1+h)-f(x_1),$$

will be equal to that of the derivatives $F'(x), f'(x)$, when in these x has some value between x_1 and x_1+h; that is, if θ_1 be a proper fraction, we shall have

$$\frac{F(x_1+h)-F(x_1)}{f(x_1+h)-f(x_1)}=\frac{F'(x_1+\theta_1 h)}{f'(x_1+\theta_1 h)}.$$

To prove this, let A be the least and B the greatest algebraic values that the fraction $\dfrac{F'(x)}{f'(x)}$ can have for values of x between x_1 and x_1+h; then the two differences,

$$\frac{F'(x)}{f'(x)}-A,\ \frac{F'(x)}{f'(x)}-B,$$

must have opposite signs for any of these values of x; and the same will be true for

$$F'(x) - Af'(x),\ F'(x) - Bf'(x),$$

because, by hypothesis, $f'(x)$ has an invariable sign between its limiting values. But these last expressions are the differential co-efficients of the two functions

$$F(x) - Af(x),\ F(x) - Bf(x).$$

One of these functions, therefore (Art. 52), must be constantly increasing, and the other constantly decreasing, while the values of x are limited by x_1 and $x_1 + h$.

If, then, the value answering to x_1 be subtracted from that answering to $x_1 + h$ for the one and the other, we have the two expressions,

$$F(x_1 + h) - F(x_1) - A\Big(f(x_1 + h) - f(x_1)\Big),$$

$$F(x_1 + h) - F(x_1) - B\Big(f(x_1 + h) - f(x_1)\Big),$$

one of which must be positive, and the other negative. Wherefore it follows, that, if both be divided by $f(x_1 + h) - f(x_1)$, the quotients

$$\frac{F(x_1 + h) - F(x_1)}{f(x_1 + h) - f(x_1)} - A,$$

$$\frac{F(x_1 + h) - F(x_1)}{f(x_1 + h) - f(x_1)} - B,$$

have opposite signs; that is, $\dfrac{F(x_1 + h) - F(x_1)}{f(x_1 + h) - f(x_1)}$ is greater than A, and less than B, and is therefore comprised between these greatest and least values of $\dfrac{F'(x)}{f'(x)}$. But $F'(x)$ and $f'(x)$ being continuous, while x passes by insensible gradations from x_1 to $x_1 + h$, the ratio $\dfrac{F'(x)}{f'(x)}$ must pass through all values in-

termediate to its greatest and least values. Hence there must be some value of x between x_1 and x_1+h that will render the ratio of the differential co-efficients equal to the ratio of the differences of the functions.

Let θ_1 be a variable proper fraction: then, from what precedes, a value may be assigned it, that, agreeably to our enunciation, will cause it to satisfy the equation

$$\frac{F(x_1+h)-F(x_1)}{f(x_1+h)-f(x_1)}=\frac{F'(x_1+\theta_1 h)}{f'(x_1+\theta_1 h}.$$

55. It has been assumed in what precedes that $f'(x)$ retains the same sign between the initial and final values of x; but the proposition is true when the assumption is made with reference to $F'(x)$, instead of $f'(x)$. For, if $F'(x)$ does not change its sign, by the same course of reasoning we can prove that

$$\frac{f(x_1+h)-f(x_1)}{F(x_1+h)-F(x_1)}=\frac{f'(x_1+\theta_1 h)}{F'(x_1+\theta_1 h)}:$$

whence

$$\frac{F(x_1+h)-F(x_1)}{f(x_1+h)-f(x_1)}=\frac{F'(x_1+\theta_1 h)}{f'(x_1+\theta_1 h)}.$$

56. From the theorem established in Art. 54, we deduce the following consequences:—

1st, If $F(x)$ and $f(x)$ both become zero for the particular value $x=x_1$, then

$$\frac{F(x_1+h)}{f(x_1+h)}=\frac{F'(x_1+\theta_1 h)}{f'(x_1+\theta_1 h)}\ldots(a).$$

2d, If the differential co-efficients up to the $(n-1)^{\text{th}}$ order of both $F(x)$ and $f(x)$ vanish for $x=x_1$, those of the second being constantly positive or constantly negative between the limits corresponding to $x=x_1$, $x=x_1+h$, while the functions themselves do not vanish for this particular value of x,

then, from what has just been proved, we shall have the following relations: —

$$\frac{F'(x_1+\theta_1 h)}{f'(x_1+\theta_1 h)}=\frac{F''(x_1+\theta_2 h)}{f''(x_1+\theta_2 h)}$$

$$\frac{F''(x_1+\theta_2 h)}{f''(x_1+\theta_2 h)}=\frac{F'''(x_1+\theta_3 h)}{f'''(x_1+\theta_3 h)}$$

. . . .

$$\frac{F^{(n-1)}(x_1+\theta_{n-1} h)}{f^{(n-1)}(x_1+\theta_{n-1} h)}=\frac{F^{(n)}(x_1+\theta_n h)}{f^{(n)}(x_1+\theta_n h)}:$$

therefore

$$\frac{F(x_1+h)-F(x_1)}{f(x_1+h)-f(x_1)}=\frac{F^{(n)}(x_1+\theta_n h)}{f^{(n)}(x_1+\theta_n h)}.$$

Since, in the reasoning, no condition has been imposed on θ_n, except that it be a proper fraction, we may omit the subscript n, and thus have

$$\frac{F(x_1+h)-F(x_1)}{f(x_1+h)-f(x_1)}=\frac{F^{(n)}(x_1+\theta h)}{f^{(n)}(x_1+\theta h)} \qquad (b).$$

If the functions reduce to zero, with their derivatives for $x=x_1$, we have

$$\frac{F(x_1+h)}{f(x_1+h)}=\frac{F^{(n)}(x_1+\theta h)}{f^{(n)}(x_1+\theta h)} \qquad (c).$$

Making the further supposition that $x_1=0$, then

$$\frac{F(h)}{f(h)}=\frac{F^{(n)}(\theta h)}{f^{(n)}(\theta h)}:$$

but, because this is true for any value of h, x may be written for h; and thus

$$\frac{F(x)}{f(x)}=\frac{F^{(n)}(\theta x)}{f^{(n)}(\theta x)} \qquad (d).$$

3d, The conditions relative to $f(x)$ that have been im-

posed in the preceding propositions are satisfied when $f(x) = (x - x_1)^n$: whence

$$f'(x) = n(x - x_1)^{n-1}$$
$$f''(x) = n(n-1)(x - x_1)^{n-2}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$f^{(n-1)}(x) = n(n-1)\ldots 2(x - x_1)$$
$$f^{(n)}(x) = 1.2.3\ldots n = f^{(n)}(x_1 + \theta h).$$

Here $f(x)$ and its successive derivatives, up to the $(n-1)^{\text{th}}$, vanish for $x = x_1$; and since, in Eq. *b*, the denominator of the first member,

$$f(x_1 + h) - f(x_1) = (x_1 - x_1 + h)^n - (x_1 - x_1)^n = h^n,$$

we have

$$F(x_1 + h) - F(x_1) = \frac{h^n}{1.2.3\ldots n} F^{(n)}(x_1 + \theta h).$$

When $n = 1$, this gives

$$F(x_1 + h) - F(x_1) = hF'(x_1 + \theta h).$$

If $F(x_1) = 0$ as well as $f(x_1) = 0$, then

$$F(x_1 + h) = \frac{h^n}{1.2.3\ldots n} F^{(n)}(x_1 + \theta h).$$

Making $x_1 = 0$, and then writing x for h in the preceding equations, they become

$$F(x) - F(0) = \frac{x^n}{1.2.3\ldots n} F^{(n)}(\theta x)$$
$$F(x) - F(0) = xF'(\theta x)$$
$$F(x) = \frac{x^n}{1.2.3\ldots n} F^{(n)}(\theta x).$$

57. The equation $\dfrac{F(x)}{f(x)} = \dfrac{F^{(n)}(\theta x)}{f^{(n)}(\theta x)}$ (Eq. *d*, Art. 56), expressed in words, enunciates the following theorem; viz.: If there be two functions, $F(x), f(x)$, which, with their differential co-efficients, are continuous, and which, with these differ-

ential co-efficients up to the $(n-1)^{\text{th}}$ order inclusively, vanish for $x=0$; and if, further, the first n differential co-efficients of one of these functions are constantly of the same sign for values of the variable between zero and another assigned value; then the ratio of the functions will be equal to that of the n^{th} differential co-efficients, when, in the latter, some intermediate value is given to the variable.

The importance of this theorem warrants us in giving it an independent demonstration.

Let $F(x)$ and $f(x)$ be two functions which vanish for $x=0$; and suppose, first, that the differential co-efficient $f'(x)$ of the second does not vanish for this value of the variable, and that it retains constantly the same sign between $x=0$ and $x=h$, which requires that $f(x)$ be continually increasing, or continually decreasing, between these limits (Art. 51), and therefore constantly positive or constantly negative, since $f(x)=0$ when $x=0$; and let A denote the least and B the greatest of the values assumed by the ratio $\dfrac{F'(x)}{f'(x)}$ for values of x between zero and h: then the two quantities, $\dfrac{F'(x)}{f'(x)}-A$, $\dfrac{F'(x)}{f'(x)}-B$, will have opposite signs; and, since $f'(x)$ does not change sign, the same will be true of the differences,

$$F'(x)-Af'(x),\ F'(x)-Bf'(x):$$

but these last are the differential co-efficients of the two functions,

$$F(x)-Af(x),\ F(x)-Bf(x),$$

one of which (Art. 51) must therefore be constantly increasing, and the other constantly decreasing; that is, since both $F(x)$ and $f(x)$ vanish for $x=0$, one must be constantly

positive and the other constantly negative between the limits answering to $x=0$, $x=h$. Therefore, because $f(x)$ is of invariable sign,

$$\frac{F(x)-Af(x)}{f(x)}=\frac{F(x)}{f(x)}-A,\quad \frac{F(x)-Bf(x)}{f(x)}=\frac{F(x)}{f(x)}-B,$$

are of opposite signs: whence it follows that the ratio of the functions is comprised between the least and the greatest values of the ratio of the differential co-efficients. But, if the variable be made to pass by insensible degrees from 0 to h, the ratio $\frac{F'(x)}{f'(x)}$, which is by hypothesis continuous, must pass through all values intermediate to A and B. If then θ denote a proper fraction, it will admit of a value such that the equation $\frac{F(x)}{f(x)}=\frac{F'(\theta x)}{f'(\theta x)}$ will be satisfied.

If the differential co-efficients of both functions, from the 1st to the $(n-1)^{\text{th}}$ orders inclusively, vanish for $x=0$, by reasoning upon them as we have upon the functions, we have

$$\frac{F'(\theta_1 x)}{f'(\theta_1 x)}=\frac{F''(\theta_2 x)}{f''(\theta_2 x)}=\frac{F'''(\theta_3 x)}{f'''(\theta_3 x)}=\ldots=\frac{F^{(n)}(\theta x)}{f^{(n)}(\theta x)}:$$

whence

$$\frac{F(x)}{f(x)}=\frac{F^{(n)}(\theta x)}{f^{(n)}(\theta x)}.$$

58. It is to be observed that the only conditions upon which the equations

$$F(x_1+h)-F(x_1)=hF'(x_1+\theta h),$$
$$F(x)-F(0)=xF'(\theta x),$$
$$F(x)=\frac{x^n}{1.2.3\ldots n}F^{(n)}(\theta x),$$

depend, are, that $F(x)$, and its differential co-efficients up to the order involved in the equations, should be continuous between assigned limits of the variable.

59. From the equation $F(x_1+h)-F(x_1)=hF'(x_1+\theta h)$ of Art. 56, it may be shown, that, if the differential co-efficient with respect to x of any expression is zero for all values of x, such expression is independent of x: for, if $F'(x)$ is zero for all values of x, the above equation becomes

$$F(x_1+h)-F(x_1)=0\,;\text{ or, } F(x_1+h)=F(x_1)\,;$$

that is, the function does not vary with, and is therefore independent of, x. It is plain, that, if the differential co-efficient is not equal to zero, the expression will vary with x. Hence those expressions only are independent of a variable for which the differential co-efficients with respect to that variable are zero for all values of the variable. And further: if two functions have the same differential co-efficient with respect to any variable, such functions can differ only by a constant; for the differential co-efficient of the function which is the difference of these functions is zero by hypothesis: therefore, by what precedes, this difference must be independent of x; that is, constant.

60. Suppose $F(x)$ to be real and continuous; then, by means of the foregoing principles, we may find the development of this function arranged according to the ascending positive powers of x.

For we have, Art. 56,

$$F(x)-F(0)=xF'(\theta x)=xF'(0)+R_1x$$

by making $$F'(\theta x)=F'(0)+R_1\,;$$

whence

$$F(x)-F(0)-xF'(0)=R_1x:$$

from which it is seen that R_1x is a quantity that reduces to zero when x is zero; and the same is true of $F'(x)-F'(0)$, which is its first derivative with respect to x. Its second deriv-

ative is $F''(x)$. Wherefore, by the article already referred to,

$$F(x) - F(0) - xF'(0) = R_1 x = \frac{x^2}{1.2} F''(\theta x).$$

Making

$$F''(\theta x) = F''(0) + R_2,$$

then, as before,

$$F(x) - F(0) - xF'(0) - \frac{x^2}{1.2} F''(0) = R_2 \frac{x^2}{1.2};$$

and it is evident that $R_2 \frac{x^2}{1.2}$ is a quantity, which, with its first and second derivatives,

$$F'(x) - F'(0) - xF''(0),\ F''(x) - F''(0),$$

vanishes with x, and that its third derivative is $F'''(x)$: therefore we have

$$F(x) - F(0) - xF'(0) - \frac{x^2}{1.2} F''(0) = \frac{x^3}{1.2.3} F'''(\theta x).$$

Next, place $F'''(\theta x) = F'''(0) + R_3$, and then proceed as before, and so on, bearing in mind that the expressions $\frac{R_3 x^3}{1.2.3}$, $\frac{R_4 x^4}{1.2.3.4} \ldots$, together with their derivatives up to the $(n-1)^{\text{th}}$ order inclusively, vanish for $x = 0$; and we should have, for our final result,

$$\left.\begin{aligned} F(x) - F(0) - xF'(0) - \frac{x^2}{1.2} F''(0) - \ldots \\ - \frac{x^{n-1}}{1.2\ldots(n-1)} F^{(n-1)}(0) \end{aligned}\right\} = \frac{x^n}{1.2\ldots n} F^{(n)}(\theta x).$$

whence

$$F(x) = F(0) + xF'(0) + \frac{x^2}{1.2} F''(0)$$
$$+ \frac{x^{n-1}}{1.2\ldots(n-1)} F^{(n-1)}(0) + \frac{x^n}{1.2.3\ldots n} F^{(n)}(\theta x).$$

And it appears that any real and continuous function $F(x)$ of x is composed of the part

$$F(0) + xF'(0) + \frac{x^2}{1.2} F''(0) + \cdots$$
$$+ \frac{x^{n-1}}{1.2.3\ldots(n-1)} F^{(n-1)}(0);$$

which is entire and rational in respect to x, and of the remainder

$$\frac{x^n}{1.2.3\ldots n}F^{(n)}(\theta x).$$

If the function is entire, and of the n^{th} degree, its derivative of the n^{th} order will be a constant; that is, $F^{(n)}(\theta x) = F^n(0)$; in which case the development will terminate with the $(n+1)^{th}$ term, and the remainder will be zero. For example, if $F(x) = (1+x)^n$, we should have

$$(1+x)^n = 1 + nx + n\frac{n-1}{2}x^2 + \ldots + nx^{n-1} + x^n.$$

61. The same principles also enable us to find the development of $F(x+h)$, arranged according to the ascending powers of either x or h. For we have, Art. 56,

$$F(x+h) - F(x) = hF'(x + \theta_1 h).$$

Make
$$hF'(x+\theta_1 h) = hF'(x) + \mathrm{R}_1,$$
then
$$F(x+h) - F(x) - hF'(x) = R_1.$$

From this it is seen that R_1 is a function of both x and h, and that it, and also its first derivative $F'(x+h) - F'(x)$ with respect to h, will vanish for $h = 0$: hence (Art. 56)

$$F(x+h) - F(x) - hF'(x) = \frac{h^2}{1.2}F''(x+\theta_2 h)$$
$$= \frac{h^2}{1.2}F''(x) + R_2$$

by making $\dfrac{h^2}{1.2}F''(x+\theta_2 h) = \dfrac{h^2}{1.2}F''(x) + R_2$:
whence

$$F(x+h) - F(x) - hF'(x) - \frac{h^2}{1.2}F''(x) = R_2;$$

and R_2, together with its first and second derivatives with respect to h, will vanish for $h = 0$: therefore

$$F(x+h) - F(x) - hF'(x) - \frac{h^2}{1.2}F''(x) = \frac{h^3}{1.2.3}F'''(x+\theta_3 h).$$

The manner of carrying on these operations is sufficiently obvious. We may write

$$\left.\begin{array}{l} F(x+h)-F(x)-hF'(x) \\ -\dfrac{h^2}{1.2}F''(x)-\ldots \\ -\dfrac{h^{(n-1)}}{1.2.3\ldots(n-1)}F^{(n-1)}(x) \end{array}\right\} = \frac{h^n}{1.2.3\ldots n}F^{(n)}(x+\theta h):$$

whence

$$F(x+h) = F(x) + hF'(x) + \frac{h^2}{1.2}F''(x) + \ldots$$
$$+ \frac{h^n}{1.2.3\ldots n}F^{(n)}(x+\theta h) \qquad (1).$$

If, in this last equation, we first make $x=0$, and then, in the result, write x for h, we find

$$F(x) = F(0) + xF'(0) + \frac{x^2}{1.2}F''(0) + \ldots$$
$$+ \frac{x^n}{1.2.3\ldots n}F^{(n)}(\theta x) \qquad (2),$$

from which it appears that the formula of Art. 60 is but a particular case of that just established. But formula 1 may also be deduced from 2. For, in 2, change x into h, and make $F(h)=f(x+h)$; then, taking the derivatives with respect to h, and in the results making $h=0$, we have

$$F'(h) = f'(x+h),$$
$$F''(h) = f''(x+h)\ldots F^{(n)}(\theta h) = f^{(n)}(x+\theta h),$$
$$F'(0) = f'(x),$$
$$F''(0) = f''(x)\ldots F^{(n)}(\theta h) = f^{(n)}(x+\theta h).$$

But, when h takes the place of x, Eq. 2 becomes

$$F(h) = F(0) + hF'(0) + \ldots + \frac{h^n}{1.2.3\ldots n}F^{(n)}(\theta h);$$

and in this, substituting the values of $F(h)$, $F(0)$, $F'(0)\ldots$ $F^n(\theta h)$, we have

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{1.2}f''(x)+\dots$$
$$+\frac{h^n}{1.2.3\dots n}f^{(n)}(x+\theta h),$$

which agrees with formula 1.

62. When $F(x)$ is such that the expression

$$\frac{x^n}{1.2.3\dots n}F^{(n)}(\theta x)$$

for values of x between assigned limits continually decreases as n increases, then, by making $n=\infty$, the formula,

$$F(x)=F(0)+xF'(0)+\frac{x^2}{1.2}F''(0)+\dots$$
$$+\frac{x^n}{1.2.3\dots n}F^{(n)}(\theta x)$$

of Art. 61 will give rise to a converging series; and it may be written

$$F(x)=F(0)+xF'(0)+\frac{x^2}{1.2}F''(0)$$
$$+\frac{x^3}{1.2.3}F'''(0)+\dots \quad (1),$$

which is Maclaurin's Formula.

So also if $F(x)$ is such that the expression

$$\frac{h^n}{1.2.3\dots n}F^{(n)}(x+\theta h)$$

for values of x between assigned limits continually decreases as n increases, then, making $n=\infty$, the formula,

$$F(x+h)=F(x)+hF'(x)+\frac{h^2}{1.2}F''(x)$$
$$+\dots\frac{h^n}{1.2.3\dots n}F^{(n)}(x+\theta h)$$

of the preceding article may be written

$$F(x+h)=F(x)+hF'(x)+\frac{h^2}{1.2}F''(x)$$
$$+\frac{h^3}{1.2.3}F'''(x)+\dots \quad (2),$$

which is Taylor's Formula.

63. It may be shown that the quantities,

$$\frac{x^n}{1.2.3\ldots n}, \quad \frac{h^n}{1.2.3\ldots n},$$

become zero when n is infinite. For take the expression

$$m(n-m+1) = m(n+1) - m^2 = 2m\,\frac{n+1}{2} - m^2$$

$$= \left(\frac{n+1}{2}\right)^2 - \left(\frac{n+1}{2}\right)^2 + 2m\,\frac{n+1}{2} - m^2$$

$$= \left(\frac{n+1}{2}\right)^2 - \left(\frac{n+1}{2} - m\right)^2,$$

which last form shows that the product $m(n-m+1)$ increases as m increases from 1 to $\frac{n+1}{2}$; that is, the product increases as the factors approach equality. This is also shown by the differential co-efficient $2\left(\frac{n+1}{2} - m\right)$ of the product taken with respect to m. Giving to m, in succession, the values $1, 2, 3, \ldots n$, the product will assume the successive values

$$n,\ 2(n-1),\ 3(n-2)\ldots(n-2)3,\ (n-1)2,\ n,$$

which increase from n up to a certain limit, and then decrease by the same gradations down to n again.

As n is the least value that this product assumes, the continued product of these results, of which there are n, will be greater than n^n; that is,

$$n.\ 2(n-1).\ 3(n-2)\ldots(n-2)3.\ (n-1)2.\ n$$
$$= \left(1.2.3\ldots(n-1)\,n\right)^2 > n^n;$$

whence

$$1.2.3\ldots(n-1)\,n > n^{\frac{n}{2}} : \therefore \frac{x^n}{1.2.3\ldots n} < \left(\frac{x}{\sqrt{n}}\right)^n.$$

But, if x is finite, $\left(\frac{x}{\sqrt{n}}\right)^n$ will be zero when n is infinite: hence

$$\frac{x^n}{1.2.3\ldots n} = 0 \text{ when } n = \infty;$$

and the same is true of $\frac{h^n}{1.2.3\ldots n}$.

Therefore it follows, that if $F^{(n)}(\theta x)$, $F^{(n)}(x + \theta h)$, are finite, the products

$$\frac{x^n}{1.2.3\ldots n} F^{(n)}(\theta x), \quad \frac{h^n}{1.2.3\ldots n} F^{(n)}(x + \theta h),$$

will diminish without limit as n is made to increase without limit; and we can, in such cases, employ Maclaurin's Formula for the development of $F(x)$, and that of Taylor for the development of $F(x+h)$, into series arranged according to the ascending powers of x for the first, and of either x or h for the second.

64. Maclaurin's Theorem, when applicable, may be stated as follows: The first term of the development of $F(x)$ is what the function becomes when $x = 0$; the second term is x multiplied by what the first differential co-efficient of the function becomes when $x = 0$; the third term is the second power of x divided by 1×2, and this quotient multiplied by what the second differential co-efficient of the function becomes when $x = 0$; and the $(n + 1)^{th}$, or general term, is the n^{th} power of x divided by the product of the natural numbers from 1 to n inclusive, and this quotient multiplied by what the n^{th} differential co-efficient of the function becomes when $x = 0$.

This theorem is of very general application for the expansion of functions of single variables, examples of which will be shortly given; but it is by no means universal: for

$$y = lx, \; y = \cot. x, \; y = a^{\frac{1}{x}},$$

are functions which become infinite when $x = 0$; and hence the first term in Maclaurin's Formula would be infinite, while the function for other values of x would be finite. There are other functions, such as $y = ax^{\frac{1}{2}}$, for which, though the functions themselves remain finite for $x = 0$, their first, or some of the following differential co-efficients, become infinite for this value of the variable; and, in such cases also, the formula would fail to give the development of the functions.

65. Taylor's Theorem may be enunciated as follows: When a function $F(x + h)$ of the algebraic sum of two variables can be developed into a series arranged according to the ascending powers of either taken as the leading variable, the first term is what the function becomes when this variable is made equal to zero; the second term is the first power of the leading variable multiplied by the first differential co-efficient of the first term taken with respect to the other variable; the third term is the second power of the leading variable divided by 1×2, and this quotient multiplied by the second differential co-efficient of the first term; and the $(n + 1)^{th}$, or general term, is the n^{th} power of the leading variable divided by the product of the natural numbers from 1 to n inclusive, and this quotient multiplied by the n^{th} differential co-efficient of the first term.

66. In Taylor's Formula, the co-efficients of the different powers of the leading variable are functions of the other variable. When one or more of these functions are such, that, for a particular value of the second variable, they become infinite, the formula fails to give the development of the original function for that value of the second variable; for then the function ceases to depend on the second variable, and is a

function of the first variable alone, and will not necessarily be infinite for the assigned value of the second variable.

For example, if we have

$$F(x) = \sqrt{x - a},$$

then $$F(x + h) = \sqrt{(x - a + h)}.$$

When $x = a$, $F(x) = 0$, and the first and all the higher differential co-efficients of $F(x)$ become infinite for this particular value of x; while, for this value, $F(x + h) = \sqrt{h}$.

It will be observed that there is a marked difference between the failing cases for Maclaurin's and Taylor's Theorems. When Maclaurin's fails for one value of the variable ($x = 0$), it fails for all; whereas Taylor's may fail for one value of the second variable, but give the true development of the function for all other values.

67. If a function becomes infinite for a finite value of the variable, its differential co-efficient will be infinite at the same time. In the case of an algebraic function, this follows from the fact that such function can become infinite for a finite value of the variable, only when it is in the form of a fraction whose denominator reduces to zero. But the denominator of a fraction never disappears in the process of differentiation: hence, if the function has a vanishing denominator, so will its differential co-efficient. In the case of transcendental functions, it is only by the examination of the different forms that the truth of this proposition can be established. Thus, in the logarithmic function $y = lx$, y becomes infinite for $x = 0$; $\frac{dy}{dx} = \frac{1}{x}$ is also infinite for this value of x; and for the exponential function $y = a^{\frac{1}{x}}$, which, if $a > 1$, becomes infinite when

$x = 0$, the differential co-efficient is $\dfrac{dy}{dx} = -\dfrac{la}{x^2} a^{\frac{1}{x}}$, which is infinite when $x = 0$.

The circular functions tan. x, cot. x, sec. x, cosec. x, which may become infinite for finite values of x, when expressed in terms of sin. x, cos. x, are fractional forms to which the reasoning in reference to algebraic functions applies.

If a function becomes infinite for an infinite value of the variable, it does not follow that the differential co-efficient becomes infinite at the same time.

Thus, in the example $y = lx$, $\dfrac{dy}{dx} = \dfrac{1}{x}$, and y is infinite when $x = \infty$; but $\dfrac{dy}{dx} = 0$ for this value of x.

68. It was remarked in Art. 62, that, unless $F(x)$ and $F(x+h)$ are such that

$$F(0) + xF'(0) + \frac{x^2}{1.2} F''(0) + \ldots,$$

$$F(x) + hF'(x) + \frac{h^2}{1.2} F''(x) + \ldots,$$

give rise to converging series, the formulas of Maclaurin and Taylor will not serve for the expansion of these functions.

A series in general is a succession of quantities any one of which is derived, according to a fixed law, from one or more of those which precede it. If $u_0, u_1, u_2, u_3, \ldots u_n$, are such quantities called the terms of the series, then we have

$$S_n = u_0 + u_1 + u_2 + u_3 + \ldots u_{n-1}$$

for the sum of the first n terms. When this sum approaches indefinitely a finite limit S, as n continually increases, the series is said to be converging, and the limit in question is called the sum of the series; but, if the sum S_n does not thus

approach any fixed limit as n increases indefinitely, the series is said to be diverging, and has no sum.

The geometrical series

$$a, ar, ar^2, \ldots ar^n,$$

having ar^n for its general term, has for its sum

$$S_n = a(1 + r + r^2 + \ldots r^n) = a\frac{1-r^n}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

It is evident that, as n increases, this sum converges towards the fixed limit $\frac{a}{1-r}$ if r is less than 1; and that, on the contrary, as n increases, the sum also increases indefinitely if r is greater than 1.

We are assured of the convergence of the series

$$u_0, u_1, u_2 \ldots u_n,$$

when, as n increases, the sum

$$S_n = u_0 + u_1 + u_2 + \ldots u_{n-1}$$

converges to a fixed limit S, and when, at the same time, the differences

$$S_{n+1} - S_n = u_n, \quad S_{n+2} - S_n = u_n + u_{n+1} \ldots,$$

vanish when n is made infinite.

The limits assigned this work do not permit an investigation of the rules by which, in many cases, the convergence or divergence of a series may be ascertained.

69. Admitting that $F(x)$ can be expanded into a series arranged according to the ascending integral powers of x, Maclaurin's Theorem may be demonstrated as follows: —

Assume

$$F(x) = A_0 + A_1 x^a + A_2 x^b + \ldots + A_n x^p$$

in which $A_0, A_1, A_2 \ldots$, do not contain x, and the exponents

$a, b, c \ldots$, are written in the order of their magnitude, a being the least; then, by successive differentiation, we have

$$F'(x) = aA_1x^{a-1} + bA_2x^{b-1} + \ldots + pA_nx^{p-1},$$

$$F''(x) = a(a-1)A_1x^{a-2} + b(b-1)A_2x^{b-2} + \ldots + p(p-1)A_nx^{p-2},$$

$$F'''(x) = a(a-1)(a-2)A_1x^{a-3} + b(b-1)(b-2)A_2x^{b-3} + \ldots + p(p-1)(p-2)A_nx^{p-3}.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The assumed and all the following equations, being true for all values of x, make $x = 0$; then, since $F(0)$, $F'(0)$, $F''(0) \ldots$, would in general reduce neither to 0 nor to ∞, we should have

$$A_0 = F(0), \quad a = 1, \quad A_1 = F'(0), \quad b = 2,$$

$$A_2 = \frac{F''(0)}{1.2}, \quad c = 3, \quad A_3 = \frac{F'''(0)}{1.2.3} \ldots$$

$$F(x) = F(0) + xF'(0) + \frac{x^2}{1.2}F''(0) + \frac{x^3}{1.2.3}F'''(0) + \ldots,$$

which is identical with the formula of Art. 62.

70. Taylor's Theorem also admits of the following simple demonstration when the function $F(x+h)$ can be expanded into a series arranged according to the ascending integral powers of one of the variables with co-efficients which are functions of the other variable only.

Assume

$$F(x+h) = f(x) + f_1(x)h^a + f_2(x)h^b + \ldots + f_n(x)h^p,$$

and differentiate with respect to x, and also with respect to h; then

$$\frac{dF(x+h)}{dx} = \frac{df(x)}{dx} + \frac{df_1(x)}{dx}h^a + \frac{df_2(x)}{dx}h^b + \ldots + \frac{df_n(x)}{dx}h^p$$

$$\frac{dF(x+h)}{dh} = af_1(x)h^{a-1} + bf_2(x)h^{b-1} + \ldots + pf_n(x)h^{p-1}.$$

But $F(x+h)$ involves h in precisely the same way that it it does x; and, if we place $x+h=y$, we have (Art. 42)

$$\frac{dF(x+h)}{dx}=\frac{dF(y)}{dy}\frac{dy}{dx}=\frac{dF(y)}{dy}\times 1,$$

$$\frac{dF(x+h)}{dh}=\frac{dF(y)}{dy}\frac{dy}{dh}=\frac{dF(y)}{dy}\times 1:$$

hence $$\frac{dF(x+h)}{dx}=\frac{dF(x+h)}{dh};$$

that is, these differential co-efficients are equal for all values of x and h, which can only be the case when they are identically the same. This requires that

$$a=1, f_1(x)=\frac{df(x)}{dx},\ b=2, f_2(x)=\frac{1}{2}\frac{df_1(x)}{dx};$$

$$c=3,\ f_3(x)=\frac{1}{2.3}\frac{df_2(x)}{dx}\ \ldots;$$

also, by making $h=0$ in the assumed development, we find

$$f(x)=F(x);$$

whence $$f_1(x)=F'(x),\ f_2(x)=\frac{F''(x)}{1.2}\ldots:$$

therefore

$$F(x+h)=F(x)+hF'(x)+\frac{h^2}{1.2}F''(x)+\ldots$$
$$+\frac{h^n}{1.2\ldots n}F^{(n)}(x).$$

SECTION VI.

EXPANSION OF FUNCTIONS.

71. THE application of the formulas, demonstrated in the preceding section for the expansion of functions, gives rise to many important series, some of which we shall now deduce.

1. If $F(x) = (1+x)^m$, then

$$F'(x) = m(1+x)^{m-1},$$

$$F''(x) = m(m-1)(1+x)^{m-2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$F^{(n-1)}(x) = m(m-1)\ldots(m-n+2)(1+x)^{m-n+1},$$

$$F^{(n)}(x) = m(m-1)\ldots(m-n+1)(1+x)^{m-n};$$

therefore $F(0) = 1$, $F'(0) = m$, $F''(0) = m(m-1)\ldots,$

$$F^{(n-1)}(0) = m(m-1)\ldots(m-n+2);$$

and hence, by Art. 60,

$$(1+x)^m = 1 + mx + m\frac{m-1}{1.2}x^2 + \ldots$$

$$+ m\frac{(m-1)\ldots(m-n+2)}{1.2.3\ldots(n-1)}x^{n-1}$$

$$\frac{m(m-1)\ldots(m-n+1)x^n}{1.2.3\ldots n}(1+\theta x)^{m-n}.$$

When x is less than unity, the last term in this development will diminish as n increases; and, by making n sufficiently great, the series

$$1 + mx + m\frac{m-1}{1.2}x^2 + m\frac{(m-1)(m-2)}{1.2.3}x^3 + \ldots$$

will approximate more and more nearly the true value of $(1+x)^m$ the greater the number of terms taken.

2. Let $F(x) = e^x$; then

$$F'(x) = e^x = F''(x) = F'''(x) = \ldots = F^{(n-1)}(x) = F^{(n)}(x),$$

$$F'(0) = 1 = F''(0) = F'''(0) = \ldots = F^{(n-1)}(0);$$

$$F^{(n)}(\theta x) = e^{\vartheta x}:$$

therefore
$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \cdots$$
$$+ \frac{x^{n-1}}{1.2.3\ldots(n-1)} + \frac{x^n}{1.2\ldots n} e^{\vartheta x}.$$

Making in this $x = 1$, we have

$$e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \cdots;$$

a series that may be used for finding the approximate value of e.

3. Let $F(x) = \sin. x$; then

$$F'(x) = \cos. x = \sin.\left(x + \frac{\pi}{2}\right),$$

$$F''(x) = \frac{d \sin.\left(x + \frac{\pi}{2}\right)}{dx} = \cos.\left(x + \frac{\pi}{2}\right) = \sin.\left(x + \frac{2\pi}{2}\right),$$

$$F'''(x) = \frac{d \sin.\left(x + \frac{2\pi}{2}\right)}{dx} = \cos.\left(x + \frac{2\pi}{2}\right) = \sin.\left(x + \frac{3\pi}{2}\right),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$F^{(n)}(x) = \sin.\left(x + \frac{n\pi}{2}\right).$$

Therefore $F(0) = 0,\ F'(0) = 1,\ F''(0) = 0,\ F'''(0) = -1,$

$$F^{(n-1)}(0) = \sin. \frac{n-1}{2}\pi;$$

and we have

$$\sin. x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2..5} + \cdots$$
$$+ \frac{x^{n-1}}{1.2.3...(n-1)} \sin. \frac{n-1}{2}\pi$$
$$+ \frac{x^n}{1.2.3...n} \sin. \left(\theta x + \frac{n\pi}{2}\right).$$

4. Let $F(x) = \cos. x$; then

$$F'(x) = -\sin. x = \cos. \left(x + \frac{\pi}{2}\right), \ F''(x) = \cos. \left(x + \frac{2\pi}{2}\right),$$
$$F'''(x) = \cos. \left(x + \frac{3\pi}{2}\right) \ldots F^{(n)}(x) = \cos. \left(x + \frac{n\pi}{2}\right),$$
$$F(0) = 1, F'(0) = 0, F''(0) = -1, F'''(0) = 0 \cdots$$
$$F^{(n-1)}(0) = \cos. \frac{n-1}{2}\pi:$$

hence
$$\cos. x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \cdots$$
$$+ \frac{x^{n-1}}{1.2.3...(n-1)} \cos. \frac{n-1}{2}\pi$$
$$+ \frac{x^n}{1.2.3...n} \cos. \left(\theta x + \frac{n\pi}{2}\right).$$

By Art. 63, it will be observed that the last terms in Exs. 2, 3, and 4, diminish as n is increased, and finally vanish when n becomes infinite.

5. Let $F(x) = l(1+x)$; then

$$F'(x) = \frac{1}{1+x}, F''(x) = -\frac{1}{(1+x)^2}, F'''(x) = \frac{1.2}{(1+x)^3} \cdots,$$
$$F^{(n)}(x) = \frac{(-1)^{n-1}\, 1.2.3...(n-1)}{(1+x)^n}: \text{ hence}$$
$$F(0) = 0, F'(x) = 1, F''(x) = -1, F'''(x) = 2 \ldots$$
$$F^{(n)}(0) = (-1)^{n-1}\, 1.2.3...(n-1); \text{ and therefore}$$
$$l(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-2}}{n-1} x^{n-1}$$
$$+ \frac{(-1)^{n-1}}{n} \frac{x^n}{(1+\theta x)^n}.$$

An examination of the last term of this expansion shows, that, when x does not exceed unity, this term necessarily decreases as n increases, and vanishes when n becomes infinite. And, since the factor $\left(\frac{x}{1+\theta x}\right)^n$ under this hypothesis cannot exceed unity, the sum of the series, up to the n^{th} term inclusive, cannot differ from the true sum by more than $\frac{1}{n}$; and hence, by increasing n sufficiently, this difference can be made as small as we please.

Changing the sign of x, we have

$$l(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{(-1)^{n-2}}{n-1}x^{n-1}$$
$$- \frac{(-1)^{n-1}}{n}\frac{x^n}{(1-\theta x)^n}.$$

6. Let $F(x) = \tan.^{-1} x$; then

$$F'(x) = \frac{1}{1+x^2} = \frac{1}{2}\left(\frac{1}{1-x\sqrt{-1}} + \frac{1}{1+x\sqrt{-1}}\right)$$
$$= \frac{1}{2}\left((1-x\sqrt{-1})^{-1} + (1+x\sqrt{-1})^{-1}\right)$$
$$F''(x) = \frac{1}{2} \times -1 \times -\sqrt{-1}\,(1-x\sqrt{-1})^{-2}$$
$$+ \frac{1}{2} \times -1 \times \sqrt{-1}\,(1+x\sqrt{-1})^{-2}$$
$$= \sqrt{-1} \times \frac{1}{2}\left((1-x\sqrt{-1})^{-2} - (-1)^2(1+x\sqrt{-1})^{-2}\right)$$
$$F'''(x)$$
$$= (\sqrt{-1})^2\,\frac{1.2}{2}\left((1-x\sqrt{-1})^{-3} - (-1)^3(1+x\sqrt{-1})^{-3}\right)$$

. . . .

$$F^{(n)}(x)$$
$$=(\sqrt{-1})^{n-1}\frac{1.2.3..(n-1)}{2}\left((1-x\sqrt{-1})^{-n} - (-1)^n(1+x\sqrt{-1})^{-n}\right):$$

therefore

$$F^{(n)}(0) = (\sqrt{-1})^{n-1} \frac{1.2.3...(n-1)}{2} \left(1 - (-1)^n\right).$$

Whence it follows, that, if n is an even number, $F^{(n)}(0) = 0$; but, if n is uneven, then

$$F^{(n)}(0) = (\sqrt{-1})^{n-1} \frac{1.2.3..(n-1)}{2} \times 2$$

$$= \frac{(\sqrt{-1})^n}{\sqrt{-1}} 1.2.3..(n-1) = \pm \frac{1}{\sqrt{-1}} 1.2.3..(n-1).$$

Hence we have

$$\tan.^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} \ldots \pm \frac{x^{n-1}}{n-1}$$

$$\mp \frac{x^n}{n} \frac{(1 - \theta x\sqrt{-1})^{-n} \mp (1 + \quad (\sqrt{-1})^{-n}}{2\sqrt{-1}}.$$

The final term in this development is not in a convenient form, as it stands, to decide whether the series is converging or diverging; but by referring to Ex. 18, p. 67, making $a = 1$, and observing that there $\theta = \frac{\pi}{2} - \tan.^{-1} x$, we have

$$F^{(n)}(x) = (-1)^{n-1} \frac{1.2.3...(n-1)}{(1+x^2)^{\frac{n}{2}}} \sin.\left(\frac{n\pi}{2} - n\tan.^{-1} x\right):$$

therefore

$$\tan.^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \&+ \ldots$$

$$(-1)^{n-1} \frac{x^n}{n} \frac{1}{(1+x^2)^{\frac{n}{2}}} \sin.\left(\frac{n\pi}{2} - n\tan.^{-1} x\right).$$

This form of the final term shows, that, if x is less than unity, the numerical value of the term may be made as small as we please by giving to n a value sufficiently great.

The above form for $F^{(n)}(x)$ might have been used for finding all the differential co-efficients of $\tan.^{-1} x$ as readily as that specially deduced for that purpose.

The following is a more simple process for getting the expansion of $\tan.^{-1} x$: —

Assume

$$\tan.^{-1} x = A + Bx + Cx^2 + Dx^3 + \&c. \quad (1),$$

and differentiate both members with respect to x; then

$$\frac{1}{1+x^2} = B + 2Cx + 3Dx^2 + \&c. \quad (2);$$

but by division, or by the Binomial Theorem,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \& + \&c. \quad (3).$$

The second members of (2) and (3) must be identical: hence, equating the co-efficients of like powers of x, we have

$$B = 1, \ C = 0, \ D = -\frac{1}{3}, \ E = 0 \ldots;$$

and, since the assumed development must be true for all values of x, make $x = 0$ in (1), and we find $A = 0$: therefore

$$\tan.^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \& - \ldots$$

7. If $y = \sin.^{-1} x$, assume

$$\sin.^{-1} x = A + Bx + Cx^2 + Dx^3 + \ldots \quad (1),$$

and differentiate both members; then

$$\frac{1}{\sqrt{1-x^2}} = B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots \quad (2);$$

but, by the Binomial Theorem, we find

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots \quad (3).$$

The co-efficients of the like powers of x in the second members of (2) and (3) must be equal: hence

$$B = 1, \ C = 0, \ D = \frac{1}{2.3}, \ E = 0, \ F = \frac{1.3}{2.4}\frac{1}{5} \ldots;$$

and, by making $x = 0$ in (1), we get $A = 0$: therefore

$$\sin.^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots$$

8. Let $y = e^{a \sin.^{-1} x}$, and assume

$$y = A_0 + A_1 x + A_2 x^2 + \ldots + A_n x^n + \ldots \quad (1).$$

Differentiate twice; then

$$\frac{dy}{dx} = A_1 + 2A_2 x + 3A_3 x^2 + \ldots + nA_n x^{n-1} + \ldots \quad (2)$$

$$\frac{d^2 y}{dx^2} = 2A_2 + 2.3A_3 x + \ldots + (n-1)nA_n x^{n-2} + \ldots \quad (3).$$

But

$$\frac{dy}{dx} = e^{a \sin.^{-1} x} \frac{a}{\sqrt{1-x^2}}$$

$$\frac{d^2 y}{dx^2} = e^{a \sin.^{-1} x} \frac{a^2}{1-x^2} + \frac{xae^{a \sin.^{-1} x}}{(1-x^2)^{\frac{3}{2}}};$$

and hence

$$(1-x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} = a^2 y \ldots \quad (4).$$

Substitute in (4) the values of $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, taken from (2) and (3), and we have

$$\left.\begin{array}{l} 2A_2 + 2.3A_3 x + 3.4A_4 x^2 + \cdots + (n-1)nA_n x^{n-2} + \ldots \\ -(2A_2 x^2 + 2.3A_3 x^3 + 3.4A_4 x^4 + \cdots + (n-1)nA_n x^n + \ldots) \\ -(\ A_1 x + 2A_2 x^2 + 3A_3 x^3 + 4A_4 x^4 + \cdots + nA_n x^n + \ldots) \end{array}\right\}$$

$$= \left\{\begin{array}{l} a^2 A_0 + a^2 A_1 x + a^2 A_2 x^2 + a^2 A_3 x^3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \quad + a^2 A_n x^n \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right.$$

Equating the co-efficients of the same powers of x in the two members of this equation, we find

$$A_2 = \frac{a^2}{2} A_0, \quad A_3 = \frac{a^2+1}{2.3} A_1, \quad A_4 = \frac{a^2+4}{3.4} A_2 \ldots,$$

and generally

$$A_n = \frac{a^2 + (n-2)^2}{(n-1)n} A_{n-2} \ldots \quad (5).$$

If then A_0 and A_1 be found, formula 5 will give all the following co-efficients in terms of these two.

A_0 is what $e^{a \sin.^{-1} x}$ becomes when $x = 0$: hence $A_0 = 1$.

And A_1 is what $\dfrac{dy}{dx} = e^{a \sin.^{-1} x} \dfrac{a}{\sqrt{1-x^2}}$ becomes when $x = 0$: hence $A_1 = a$.

In formula 5, making n equal to 2, 3, 4, &c., successively, we get

$$A_2 = \frac{a^2}{1.2}, \; A_3 = \frac{(a^2+1)a}{1.2.3}, \; A_4 = \frac{(a^2+2^2)a^2}{1.2.3.4} \dots$$

Substituting these values of $A_0, A_1, A_2 \dots$ in the assumed development, it becomes

$$e^{a \sin.^{-1} x} = 1 + ax + \frac{a^2x^2}{1.2} + \frac{a(a^2+1)}{1.2.3}x^3 + \frac{a^2(a^2+2^2)}{1.2.3.4}x^4$$
$$+ \frac{a(a^2+1)(a^2+3^2)}{1.2...5}x^5 + \dots$$

By Ex. 2 of this article, we also have

$$e^{a \sin.^{-1} x} = 1 + a \sin.^{-1} x + \frac{a^2}{1.2}(\sin.^{-1} x)^2 + \frac{a^3}{1.2.3}(\sin.^{-1} x)^3 + \dots$$

Equating the co-efficient of the first power of a in this series with that of the same power of a in the preceding series, we have

$$\sin.^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \&c.,$$

as in Ex. 7. By equating the co-efficients of a^2, we should also find

$$(\sin.^{-1} x)^2 = x^2 + \frac{2^2}{3.4}x^4 + \frac{2^2.4^2}{3.4.5.6}x^6 + \frac{2^2.4^2.6^2.}{3.4.5.6.7.8}x^8 \dots$$

9. $y = l(1+e^x)$, $\qquad y = l2 + \dfrac{x}{2} + \dfrac{x^2}{2^3} - \dfrac{x^4}{2^3.1.2.3.4} + \dots$

10. $y = l(1 - x + x^2)$, $y = -x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} \dots$

11. $y = l(1 + \sin. x)$, $y = x - \frac{x^2}{2} + \frac{x^3}{6} \dots$

12. $y = e^{\tan.^{-1} x}$, $y = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24} \dots$

13. If $y = \left(\frac{(x-a)^7 (x-b)^{10}}{(x-c)^2}\right)^{\frac{1}{5}}$, show for what values of x Taylor's Theorem fails to give the development.

It fails for $x = c$; 1st term is then infinite.

It fails for $x = a$; 2d differential co-efficient is then infinite.

SECTION VII.

APPLICATION OF SOME OF THE PRECEDING SERIES TO TRIGONOMETRICAL AND LOGARITHMIC EXPRESSIONS.

72. LET a and b represent any two real quantities whatever; then $a + b\sqrt{-1}$ will be the most general symbol for quantity, since, by giving to a and b suitable values, it may be made to embrace every conceivable quantity, real or imaginary.

The two expressions, $a + b\sqrt{-1}$, $a - b\sqrt{-1}$, which differ only in the signs of their second terms, are said to be conjugate; and their product, $(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 + b^2$, is always real and positive. The numerical value of the square root of $a^2 + b^2$ is the *modulus* of either of the conjugate expressions. Denote this modulus by r; then it may be shown that the expression $a + b\sqrt{-1}$ can be put under the form

$$r(\cos.\theta + \sqrt{-1}\sin.\theta).$$

For let $a = r\cos.\theta$, $b = r\sin.\theta$:

$$\therefore \quad \tan.\theta = \frac{b}{a}, \quad r^2(\cos.^2\theta + \sin.^2\theta) = r^2 = a^2 + b^2,$$

$$r = \sqrt{a^2 + b^2}.$$

Now, if we suppose the arc of a circle to start from $-\frac{\pi}{2}$, and to increase by continuous degrees to $+\frac{\pi}{2}$, passing through zero, the tangent will at the same time increase by continuous degrees, and pass through all possible values between $-\infty$ and $+\infty$. Among these values of the tangent, there must be one that will satisfy the equation $\tan.\theta = \frac{b}{a}$; and the arc an-

swering to this tangent will be that whose sine and cosine will satisfy the equations $a = r\cos.\theta$, $b = r\sin.\theta$, and therefore render $r(\cos.\theta + \sqrt{-1}\sin.\theta)$ the equivalent of $a + b\sqrt{-1}$.

73. Let us resume the series (Art. 71, Exs. 2, 3, 4).

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \cdots \quad (1)$$

$$\sin.x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \cdots \quad (2)$$

$$\cos.x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \cdots \quad (3)$$

and in (1) write $x\sqrt{-1}$, $-x\sqrt{-1}$, for x successively; then

$$e^{x\sqrt{-1}} = 1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} + \frac{x^4}{1.2.3.4}$$
$$+ \frac{x^5\sqrt{-1}}{1.2.3.4.5} + \cdots = \cos.x + \sqrt{-1}\sin.x,$$

as is seen by comparing this result with the second members of (2) and (3).

Also $e^{-x\sqrt{-1}} = 1 - x\sqrt{-1} - \dfrac{x^2}{1.2} + \dfrac{x^3\sqrt{-1}}{1.2.3.} + \dfrac{x^4}{1.2.3.4}$

$$- \frac{x^5\sqrt{-1}}{1.2.3.4.5} + \cdots = \cos.x - \sqrt{-1}\sin.x:$$

therefore $\quad \cos.x + \sqrt{-1}\sin.x = e^{x\sqrt{-1}} \cdots \quad (4)$

$\quad \cos.x - \sqrt{-1}\sin.x = e^{-x\sqrt{-1}} \cdots (5),$

also $\quad \cos.y + \sqrt{-1}\sin.y = e^{y\sqrt{-1}} \cdots \quad (6);$

multiplying (4) by (6)

$$(\cos.x + \sqrt{-1}\sin.x)(\cos.y + \sqrt{-1}\sin.y) = e^{(x+y)\sqrt{-1}}$$
$$= \cos.(x+y) + \sqrt{-1}\sin.(x+y).$$

Effecting the multiplication in the first member, and then equating the real part in one member with the real part in the

other, and the imaginary part in the one with the imaginary part in the other, we find

$$\cos.(x+y) = \cos.x\cos.y - \sin.x\sin.y$$
$$\sin.(x+y) = \sin.x\cos.y + \sin.y\cos.x.$$

Again:

$$\left(\cos.x+\sqrt{-1}\sin.x\right)\left(\cos.y+\sqrt{-1}\sin.y\right)\left(\cos.z+\sqrt{-1}\sin.z\right)\cdots$$
$$= e^{(x+y+z\ldots)\sqrt{-1}} = \cos.(x+y+z+\cdots)+\sqrt{-1}\sin.(x+y+z+\cdots),$$

from which, by making $x = y = z = \cdots$, we have

$$\left(\cos.x+\sqrt{-1}\sin.x\right)^m = \cos.mx+\sqrt{-1}\sin.mx,$$

and generally

$$\left(\cos.x\pm\sqrt{-1}\sin.x\right)^m = \cos.mx\pm\sqrt{-1}\sin.mx,$$

which is known as De Moivre's Formula.

Hence the multiplication of expressions of the form of $\cos.x+\sqrt{-1}\sin.x$, and therefore of all imaginary expressions, is thus reduced to an addition, and the raising to powers to a multiplication.

74. Dividing formula (4) of the preceding article by (5) of the same, member by member, we have

$$\frac{e^{x\sqrt{-1}}}{e^{-x\sqrt{-1}}} = e^{2x\sqrt{-1}} = \frac{\cos.x+\sqrt{-1}\sin.x}{\cos.x-\sqrt{-1}\sin.x} = \frac{1+\sqrt{-1}\tan.x}{1-\sqrt{-1}\tan.x};$$

whence, by taking the Napierian logarithms of both members,

$$2x\sqrt{-1} = l\left(1+\sqrt{-1}\tan.x\right) - l\left(1-\sqrt{-1}\tan.x\right).$$

Expanding the terms in the second member by Ex. 5, Art. 71,

$$2x\sqrt{-1} = \sqrt{-1}\tan.x + \frac{\tan.^2x}{2} - \sqrt{-1}\frac{\tan.^3x}{3} - \frac{\tan.^4x}{4}$$
$$+\sqrt{-1}\frac{\tan.^5x}{5} + \cdots$$
$$-\left(-\sqrt{-1}\tan.x + \frac{\tan.^2x}{2} + \sqrt{-1}\frac{\tan.^3x}{3}\right.$$
$$\left.-\frac{\tan.^4x}{4} - \sqrt{-1}\frac{\tan.^5x}{5} + \cdots\right).$$

Equating the imaginary parts in the two members of this equation, and then dividing through by $2\sqrt{-1}$, we have

$$x = \tan. x - \frac{\tan.^3 x}{3} + \frac{\tan.^5 x}{5} - \frac{\tan.^7 x}{7} + \& - \&c.,$$

a series that may be used for the calculation of π, and which agrees with the formula in Ex. 6, Art. 71.

75. To find the expansion of $\cos.^n x$ in terms of the cosines of multiples of x.

Make $\quad e^{x\sqrt{-1}} = y;$ then $e^{mx\sqrt{-1}} = y^m,\ e^{-x\sqrt{-1}} = \frac{1}{y},$

$$e^{-mx\sqrt{-1}} = \frac{1}{y^m}.$$

From formulas 4, 5, Art. 73, we find

$$2\cos. x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = y + \frac{1}{y}$$

$$2\sqrt{-1}\sin. x = e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = y - \frac{1}{y};$$

also, from De Moivre's Theorem, we deduce

$$2\cos. mx = y^m + \frac{1}{y^m},\quad 2\sqrt{-1}\sin. mx = y^m - \frac{1}{y^m}.$$

Because $\quad 2\cos. x = y + \frac{1}{y},\quad 2^n \cos.^n x = \left(y + \frac{1}{y}\right)^n;$

but

$$\left(y + \frac{1}{y}\right)^n = y^n + ny^{n-2} + n\frac{n-1}{1.2}y^{n-4} + \cdots$$

$$\cdots + n\frac{n-1}{1.2}\frac{1}{y^{n-4}} + n\frac{1}{y^{n-2}} + \frac{1}{y^n},$$

$$= y^n + \frac{1}{y^n} + n\left(y^{n-2} + \frac{1}{y^{n-2}}\right) + \cdots$$

$$n\frac{n-1}{1.2}\left(y^{n-4} + \frac{1}{y^{n-4}}\right) + \cdots \quad (a),$$

by combining terms at equal distances from the extremes: hence

$$\cos.^n x = \frac{1}{2^{n-1}}\Big(\cos. nx + n\cos.(n-2)x$$
$$+ n\frac{n-1}{1.2}\cos.(n-4)x + \cdots\Big) \quad (b).$$

Since there are $n+1$ terms in series (a), when n is even, the number of terms is odd, and the middle term, that is,

$$\frac{n(n-1)(n-2)\ldots\left(\frac{n}{2}+2\right)\left(\frac{n}{2}+1\right)}{1.2.3.\ldots\left(\frac{n}{2}-1\right)\frac{n}{2}},$$

will be independent of y, and consequently of x; but, when n is odd, $n+1$ is even, and there is no middle term in series (a), and therefore no term independent of x. In the first case, there will be within the ( ) in formula (b), besides the term that does not depend on x, $\frac{n}{2}$ terms, containing as factors the first $\cos. nx$, the second $\cos.(n-2)x$; and so on to the last, which will have $\cos. 2x$ for a factor. In the second case, that is, when n is odd, there is no term within the ( ) in formula (b) that does not involve x; but the $\frac{n+1}{2}$ terms will then have for factors, severally, $\cos. nx$, $\cos.(n-2)x\ldots$, $\cos. 3x$, $\cos. x$.

Ex. 1. $$\cos.^4 x = \frac{1}{2^3}\left(\cos. 4x + 4\cos. 2x + 6\right).$$

Ex. 2. $$\cos.^5 x = \frac{1}{2^4}\left(\cos. 5x + 5\cos. 3x + 10\cos. x\right).$$

76. To find the expansion of $\sin.^n x$ in terms of the sines of multiples of x.

By formulas 4 and 5 of Art. 73, we have, employing the notation of the last article,

$$2\sqrt{-1}\sin. x = e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = y - \frac{1}{y}:$$

therefore $\quad 2^n(\sqrt{-1})^n \sin.^n x = 2^n(-1)^{\frac{n}{2}} \sin.^n x = \left(y - \frac{1}{y}\right)^n$

$$= y^n - ny^{n-2} + n\frac{n-1}{1.2}y^{n-4} - \&+\cdots$$

$$+(-1)^{n-2}n\frac{n-1}{1.2}\frac{1}{y^{n-4}} + (-1)^{n-1}n\frac{1}{y^{n-2}} + (-1)^n\frac{1}{y^n}\cdots$$

$$= \left(y^n + (-1)^n\frac{1}{y^n}\right) - n\left(y^{n-2} - (-1)^{n-1}\frac{1}{y^{n-2}}\right)$$

$$+\frac{n(n-1)}{1.2}\left(y^{n-4} + (-1)^{n-2}\frac{1}{y^{n-4}}\right) - \&+\cdots (a).$$

An examination of this series shows, that, when n is even, the second terms within the ( ) are all plus; and, when n is odd, they are all minus. In the first case, the expansion of $\sin.^n x$ will involve only the cosines of multiples of x; and, in the second case, it will involve only the sines of these multiples.

The factor $(-1)^{\frac{n}{2}}$ in the first member will be positive and real when n is any one of the alternate even numbers beginning with 0; that is, when n is 0 or 4 or 8 or 12, &c.; and negative and real when n is one of the alternate even numbers beginning with 2. In like manner, $(-1)^{\frac{n}{2}}$ will be imaginary and positive when n is any one of the alternate odd numbers beginning with 1; and it will be imaginary and negative when n is any other odd number.

Let k represent any positive whole number, zero included; then the different series of values above indicated for n will be embraced in the four forms, $4k$, $4k+2$; $4k+1$, $4k+3$.

It would be of no advantage to make formula (a) conform to each of these cases by special notation, as it can be easily applied, as it now stands, to the examples falling under it.

Ex. 1. Expand $\sin.^3 x$ in terms of the sines of the multiples of x.

$$2^3(\sqrt{-1})^3\sin.^3 x = \left(y^3 - \frac{1}{y^3}\right) - 3\left(y - \frac{1}{y}\right)$$

$$= 2\sqrt{-1}\sin. 3x - 6\sqrt{-1}\sin. x:$$

$$\therefore \qquad \sin.^3 x = -\frac{1}{2^2}(\sin. 3x - 3\sin. x).$$

Ex. 2. Expand $\sin.^4 x$ in terms of the cosines of the multiples of x.

$$2^4(\sqrt{-1})^4\sin.^4 x = \left(y^4 + \frac{1}{y^4}\right) - 4\left(y^2 + \frac{1}{y^2}\right) + 12$$

$$= 2\cos. 4x - 8\cos. 2x + 12:$$

$$\therefore \qquad \sin.^4 x = \frac{1}{2^3}\left(\cos. 4x - 4\cos. 2x + 6\right).$$

Ex. 3. Expand $\sin.^5 x$ in terms of the sines of the multiples of x.

$$2^5(\sqrt{-1})^5\sin.^5 x = \left(y^5 - \frac{1}{y^5}\right) - 5\left(y^3 - \frac{1}{y^3}\right) + 10\left(y - \frac{1}{y}\right)$$

$$= 2\sqrt{-1}\sin. 5x - 10\sqrt{-1}\sin. 3x + 20\sqrt{-1}\sin. x:$$

$$\therefore \qquad \sin.^5 x = \frac{1}{2^4}\left(\sin. 5x - 5\sin. 3x + 10\sin. x\right).$$

Ex. 4. Expand $\sin.^6 x$ in terms of the cosines of the multiples of x.

$$2^6(\sqrt{-1})^6\sin.^6 x = \left(y^6 + \frac{1}{y^6}\right) - 6\left(y^4 + \frac{1}{y^4}\right) + 15\left(y^2 + \frac{1}{y^2}\right) - 20$$

$$= 2\cos. 6x - 12\cos. 4x + 30\cos. 2x - 20:$$

$$\therefore \qquad \sin.^6 x = -\frac{1}{2^5}\left(\cos. 6x - 6\cos. 4x + 15\cos. 2x - 10\right).$$

77. To find the different n^{th} roots of unity.

Let x represent the general value of the n^{th} root of unity; then, by the definition of the root of a number, $x^n = 1$, or

$x^n - 1 = 0$; and the object of the investigation is to find all of the values of x that will satisfy the equation $x^n - 1 = 0$.

By De Moivre's Theorem, Art. 73, we have

$$(\cos. y \pm \sqrt{-1} \sin. y)^m = \cos. my \pm \sqrt{-1} \sin. my;$$

an equation which holds, whether m is entire or fractional, positive or negative. Now, if k be any whole number, $2k\pi$ will be an exact number of circumferences to the radius unity, and

$$\cos. (y + 2k\pi) = \cos. y, \quad \sin. (y + 2k\pi) = \sin. y:$$

therefore
$$(\cos. y \pm \sqrt{-1} \sin. y)^m$$
$$= \left(\cos. (y + 2k\pi) \pm \sqrt{-1} \sin. (y + 2k\pi)\right)^m.$$

Make $m = \frac{1}{n}$; then $(\cos. y \pm \sqrt{-1} \sin. y)^{\frac{1}{n}}$

$$= \left(\cos. (y + 2k\pi) \pm \sqrt{-1} \sin. (y + 2k\pi)\right)^{\frac{1}{n}}$$

$$= \cos. \frac{y + 2k\pi}{n} \pm \sqrt{-1} \sin. \frac{y + 2k\pi}{n}.$$

In this last equation, make $y = 0$: whence, as $\cos. 2k\pi = 1$, and $\sin. 2k\pi = 0$, we have

$$(1)^{\frac{1}{n}} = \cos. \frac{2k\pi}{n} \pm \sqrt{-1} \sin. \frac{2k\pi}{n}.$$

But, from the equation $x^n - 1 = 0$, we get $x = (1)^{\frac{1}{n}}$: hence we conclude that the different values of x, or the roots of the equation $x^n - 1 = 0$, are the values that may be assumed by

$$\cos. \frac{2k\pi}{n} \pm \sqrt{-1} \sin. \frac{2k\pi}{n}$$

by assigning different values to k. Since k may be any whole number, take for it successively 0, 1, 2, &c.; then,

when $\qquad k = 0, \quad 1^{\frac{1}{n}} = \cos. 0 \pm \sqrt{-1} \sin. 0 = 1,$

when $\qquad k = 1, \quad 1^{\frac{1}{n}} = \cos. \frac{2\pi}{n} \pm \sqrt{-1} \sin. \frac{2\pi}{n},$

when $\quad k=2,\ 1^{\frac{1}{n}}=\cos.\dfrac{4\pi}{n}\pm\sqrt{-1}\sin.\dfrac{4\pi}{n},$

. . . . ;

and so on, continuing the substitutions for k until the arc $\dfrac{2k\pi}{n}$ reaches such a value as to cause the expression $\cos.\dfrac{2k\pi}{n}\pm\sqrt{-1}\sin.\dfrac{2k\pi}{n}$ to reproduce the roots it has already given. When n is an even number, this will be the case for $k=\dfrac{n}{2}$; for,

if $\quad k=\dfrac{n}{2}-1,\ 1^{\frac{1}{n}}=\cos.\dfrac{n-2}{n}\pi\pm\sqrt{-1}\sin.\dfrac{n-2}{n}\pi;$

if $\quad k=\dfrac{n}{2},\qquad 1^{\frac{1}{n}}=\cos.\pi\pm\sqrt{-1}\sin.\pi\quad=-1;$

if $\quad k=\dfrac{n}{2}+1,\ 1^{\frac{1}{n}}=\cos.\dfrac{n+2}{n}\pi\pm\sqrt{-1}\sin.\dfrac{n+2}{n}\pi;$

but
$$\cos.\frac{n-2}{n}\pi\pm\sqrt{-1}\sin.\frac{n-2}{n}\pi$$
$$=\cos.\frac{n+2}{n}\pi\mp\sqrt{-1}\sin.\frac{n+2}{n}\pi:$$

therefore the two roots corresponding to $k=\dfrac{n}{2}+1$ are the same as those corresponding to $k=\dfrac{n}{2}-1$. So, also, those obtained by substituting $\dfrac{n}{2}+2$ for k are equal to those obtained by substituting $\dfrac{n}{2}-2$ for k, and so on: whence all substitutions for k after the value $\dfrac{n}{2}$ would merely reproduce the roots already found.

Again: when n is an odd number, the substitutions for k must be continued until $k=\dfrac{n-1}{2}$; for,

if $k = \frac{n-1}{2}$, $1^{\frac{1}{n}} = \cos.\frac{n-1}{n}\pi \pm \sqrt{-1}\sin.\frac{n-1}{n}\pi$;

if $k = \frac{n-1}{2} + 1$, $1^{\frac{1}{n}} = \cos.\frac{n+1}{n}\pi \pm \sqrt{-1}\sin.\frac{n+1}{n}\pi$;

but $\cos.\frac{n-1}{n}\pi \pm \sqrt{-1}\sin.\frac{n-1}{n}\pi = \cos.\frac{n+1}{n}\pi$

$$\mp \sqrt{-1}\sin.\frac{n+1}{n}\pi:$$

hence the substitutions of $\frac{n-1}{2}$ and $\frac{n+1}{2}$ for k give the same roots. So, also, it may be shown that the substitutions of $\frac{n-3}{2}$ and $\frac{n+3}{2}$ for k would give the same roots. Therefore we should merely reproduce the roots already found, if we substituted values for k greater than $k = \frac{n-1}{2}$.

When n is even, $k = 0$ and $k = \frac{n}{2}$ give, the first the root $+1$, and the second the root -1; and the intermediate values of k give each two roots. When n is odd, $k = 0$ gives the root 1; and all the other values of k, up to $\frac{n-1}{2}$ inclusively, give each two roots. In either case, the expression $\cos.\frac{2k\pi}{n} \pm \sqrt{-1}\sin.\frac{2k\pi}{n}$ can assume n different values, and no more. Hence it follows that the equation $x^n - 1 = 0$ has n different roots, and can have no more.

By the aid of the foregoing principles, the roots of the equation $x^n - 1 = 0$ may be expressed under the form of exponentials.

Since, by Eqs. 4, 5, Art. 73, we have

$$\cos. x \pm \sqrt{-1}\sin. x = e^{\pm x\sqrt{-1}},$$

the successive values taken by the expression

$$\cos.\frac{2k\pi}{n} \pm \sqrt{-1}\sin.\frac{2k\pi}{n}$$

may be represented in order by

$$e^{\pm 0\sqrt{-1}},\ e^{\pm\frac{2\pi}{n}\sqrt{-1}},\ e^{\pm\frac{4\pi}{n}\sqrt{-1}} \ldots,\ e^{\pm\pi\sqrt{-1}},$$

when n is even, and by

$$e^{+0\sqrt{-1}},\ e^{+\frac{2\pi}{n}\sqrt{-1}},\ e^{\pm\frac{4\pi}{n}\sqrt{-1}} \ldots,\ e^{\pm\frac{n-1}{n}\pi\sqrt{-1}},$$

when n is odd; the first term in each series of roots being unity, but the last term in the first series is minus 1, since it is equal to $\cos.\pi \pm \sqrt{-1}\sin.\pi = -1$. Both series of roots are the terms of a geometrical progression, the first term of which is $e^{\pm 0\sqrt{-1}} = 1$, and of which the ratio is $e^{\pm\frac{2\pi}{n}\sqrt{-1}}$.

Ex. 1. What are the three cube-roots of unity?

They are the roots of the equation $x^3 - 1 = 0$.

Here $n = 3$, and the proper values for k in the expression $\cos.\frac{2k\pi}{n} \pm \sqrt{-1}\sin.\frac{2k\pi}{n}$ are 0 and 1: hence the first gives

$$(1)^{\frac{1}{3}} = \cos.0 \pm \sqrt{-1}\sin.0 = e^{\pm 0\sqrt{-1}} = 1.$$

The second gives

$$(1)^{\frac{1}{3}} = \cos.\frac{2\pi}{3} \pm \sqrt{-1}\sin.\frac{2\pi}{3} = e^{\pm\frac{2\pi}{3}\sqrt{-1}}.$$

Ex. 2. Find the roots of $x^6 - 1 = 0$.

Here $n = 6$, and the proper values for k are 0, 1, 2, 3.

$$(1)^{\frac{1}{6}} = \cos.0 \pm \sqrt{-1}\sin.0 = e^{\pm 0\sqrt{-1}} = 1.$$

$$(1)^{\frac{1}{6}} = \cos.\frac{\pi}{3} \pm \sqrt{-1}\sin.\frac{\pi}{3} = e^{\pm\frac{\pi}{3}\sqrt{-1}}.$$

$$(1)^{\frac{1}{6}} = \cos.\frac{2\pi}{3} \pm \sqrt{-1}\sin.\frac{2\pi}{3} = e^{\pm\frac{2\pi}{3}\sqrt{-1}}.$$

$$(1)^{\frac{1}{6}} = \cos.\pi \pm \sqrt{-1}\sin.\pi = e^{\pm\pi\sqrt{-1}} = -1.$$

For each root of the equation $x^n - 1 = 0$, there is a binomial factor of the first degree with respect to x in the first member of the equation. Since $k = 0$ gives but one root, unity, there will be but one corresponding factor $x - 1$: $k = 1$ gives two roots, and the corresponding factors are

$$x - \left(\cos. \frac{2\pi}{n} + \sqrt{-1} \sin. \frac{2\pi}{n}\right),\ x - \left(\cos. \frac{2\pi}{n} - \sqrt{-1} \sin. \frac{2\pi}{n}\right),$$

which by multiplication will produce the quadratic factor

$$x^2 - 2x \cos. \frac{2\pi}{n} + 1.$$

In like manner, each pair of simple factors may be reduced to a quadratic factor. If n is even, the last factor is $x + 1$, which may be combined with the first factor $x - 1$, producing the quadratic factor $x^2 - 1$. Hence, when n is even, we have

$$x^n - 1 = (x^2 - 1)\left(x^2 - 2x \cos. \frac{2\pi}{n} + 1\right)\left(x^2 - 2x \cos. \frac{4\pi}{n} + 1\right) \cdots \left(x^2 - 2x \cos. \frac{n-2}{n}\pi + 1\right);$$

and, when n is odd,

$$x^n - 1 = (x - 1)\left(x^2 - 2x \cos. \frac{2\pi}{n} + 1\right)\left(x^2 - 2x \cos. \frac{4\pi}{n} + 1\right) \cdots \left(x^2 - 2x \cos. \frac{n-1}{n}\pi + 1\right).$$

Ex. 1. $(x^3 - 1) = (x - 1)\left(x^2 - 2x \cos. \frac{2\pi}{3} + 1\right)$.

Ex. 2. $(x^6 - 1)$

$$= (x^2 - 1)\left(x^2 - 2x \cos. \frac{\pi}{3} + 1\right)\left(x^2 - 2x \cos. \frac{2\pi}{3} + 1\right).$$

78. The solution of the equation $x^n + 1 = 0$, and the resolution of its first member into factors.

Resume the equation

$$(\cos. y \pm \sqrt{-1} \sin. y)^{\frac{1}{n}} = \cos. \frac{y + 2k\pi}{n} \pm \sqrt{-1} \sin. \frac{y + 2k\pi}{n}$$

of Art. 77, and make $y = \pi$; then, since $\cos. \pi = -1$, and $\sin. \pi = 0$, this equation becomes

$$(-1)^{\frac{1}{n}} = \cos. \frac{2k+1}{n}\pi \pm \sqrt{-1}\sin. \frac{2k+1}{n}\pi.$$

But, from $x^n + 1 = 0$, we have $x = (-1)^{\frac{1}{n}}$; hence the roots of the equation $x^n + 1 = 0$ are the values of which the expression $\cos. \frac{2k+1}{n}\pi \pm \sqrt{-1}\sin. \frac{2k+1}{n}\pi$ will admit for admissible values of k. But k may be any whole number including zero. Therefore take for k successively the values $0, 1, 2 \ldots$; then,

$$\text{for } k = 0,\ (-1)^{\frac{1}{n}} = \cos. \frac{\pi}{n} \pm \sqrt{-1}\sin. \frac{\pi}{n};$$

$$\text{for } k = 1,\ (-1)^{\frac{1}{n}} = \cos. \frac{3\pi}{n} \pm \sqrt{-1}\sin. \frac{3\pi}{n};$$

. . . .

$$\text{for } k = \frac{n}{2} - 1,\ (-1)^{\frac{1}{n}} = \cos. \frac{n-1}{n}\pi \pm \sqrt{-1}\sin. \frac{n-1}{n}\pi.$$

When n is even, substitutions for k greater than $\frac{n}{2} - 1$ will only reproduce preceding values for $(-1)^{\frac{1}{n}}$; for, if $k = \frac{n}{2}$, then

$$(-1)^{\frac{1}{n}} = \cos. \left(\pi + \frac{\pi}{n}\right) \pm \sqrt{-1}\sin. \left(\pi + \frac{\pi}{n}\right)$$

$$= \cos. \left(\pi - \frac{\pi}{n}\right) \mp \sqrt{-1}\sin. \left(\pi - \frac{\pi}{n}\right),$$

which is the same pair of roots as that given by the substitution of $\frac{n}{2} - 1$ for k. In like manner, it may be shown, that, if $k = \frac{n}{2} + 1$, the pair of roots would be the same as that for $k = \frac{n}{2} - 2$; and so on.

When n is odd, the substitutions for k must be continued until $k = \frac{n-1}{2}$; for,

if $k = \frac{n-3}{2}$, $(-1)^{\frac{1}{n}} = \cos. \frac{n-2}{n}\pi \pm \sqrt{-1}\sin. \frac{n-2}{n}\pi$;

if $k = \frac{n-1}{2}$, $(-1)^{\frac{1}{n}} = \cos. \pi \pm \sqrt{1}\sin. \pi = -1$.

Now, for the next value of k, that is, $k = \frac{n-1}{2} + 1 = \frac{n+1}{2}$,

$$(-1)^{\frac{1}{n}} = \cos. \frac{n+2}{n}\pi \pm \sqrt{-1}\sin. \frac{n+2}{n}\pi$$

$$= \cos. \frac{n-2}{n}\pi \mp \sqrt{-1}\sin. \frac{n-2}{n}\pi;$$

and therefore this substitution for k gives the same pair of roots as is given for $k = \frac{n-3}{2}$, and the higher values of k merely cause preceding pairs of roots to recur. Hence, whether n be even or odd, there will be n, and only n, different values for $(-1)^{\frac{1}{n}}$; and the equation $x^n + 1 = 0$ has n, and only n, different roots. These roots can be put under the form of exponentials, as in the case of the roots of $x^n - 1 = 0$.

Ex. 1. What are the roots of $x^4 + 1 = 0$?

Here $n = 4$; and the formula

$$(-1)^{\frac{1}{n}} = \cos. \frac{2k+1}{n}\pi \pm \sqrt{-1}\sin. \frac{2k+1}{n}\pi$$

gives, for $k = 0$, $(-1)^{\frac{1}{n}} = \cos. \frac{\pi}{4} \pm \sqrt{-1}\sin. \frac{\pi}{4}$;

for $k = 1$, $(-1)^{\frac{1}{n}} = \cos. \frac{3\pi}{4} \pm \sqrt{-1}\sin. \frac{3\pi}{4}$.

Ex. 2. What are the roots of $x^5+1=0$?

Here $n=5$; and the formula gives,

$$\text{for } k=0, \quad (-1)^{\frac{1}{n}} = \cos.\frac{\pi}{5} \pm \sqrt{-1}\sin.\frac{\pi}{5};$$

$$\text{for } k=1, \quad (-1)^{\frac{1}{n}} = \cos.\frac{3\pi}{5} \pm \sqrt{-1}\sin.\frac{3\pi}{5};$$

$$\text{for } k=2, \quad (-1)^{\frac{1}{n}} = \cos.\pi \pm \sqrt{-1}\sin.\pi = -1.$$

For each root of the equation $x^n+1=0$, there is a corresponding binomial factor of the first degree with respect to x in the first member of the equation.

When n is even, all the roots enter the equation by conjugate pairs; and the factors of the first member, answering to the simple roots of each pair, may be compounded into a rational quadratic factor, and we should have

$$x^n+1 = \left(x^2 - 2x\cos.\frac{\pi}{n}+1\right)\left(x^2-2x\cos.\frac{3\pi}{n}+1\right)\cdots$$
$$\cdots\left(x^2-2x\cos.\frac{n-1}{n}\pi+1\right).$$

When n is odd, there will be rational quadratic factors for $k=0$, $k=1\ldots$, up to $k=\dfrac{n-3}{n}$ inclusively; but, for $k=\dfrac{n-1}{2}$, there is only the simple factor $x+1$; so that, in this case, we should have

$$x^n+1 = \left(x^2 - 2x\cos.\frac{\pi}{n}+1\right)\left(x^2-2x\cos.\frac{3\pi}{n}+1\right)\cdots$$
$$\cdots\left(x^2-2x\cos.\frac{n-2}{n}\pi+1\right)(x+1).$$

The solution of the equations $x^n-a=0$, $x^n+a=0$, and the resolution of their first members into simple and quadratic

factors, may be at once effected by the formulas in this and the preceding articles: for these equations give respectively

$$x = a^{\frac{1}{n}}(1)^{\frac{1}{n}}, \quad x = (a)^{\frac{1}{n}}(-1)^{\frac{1}{n}},$$

in both of which $a^{\frac{1}{n}}$ is the numerical value of the n^{th} root of a; and this, multiplied by the different values of $(1)^{\frac{1}{n}}$, will give the roots of $x^n - a = 0$; and, multiplied by the values of $(-1)^{\frac{1}{n}}$, will give the roots of $x^n + a = 0$.

79. The determination of a general expression for the logarithm of a number positive or negative.

In any system of logarithms, the logarithm of 1 is 0, and the logarithm of 0 is $-\infty$ if the base is greater than unity, and $+\infty$ if the base is less than unity; while the logarithm of ∞ is $+\infty$ or $-\infty$, according as the base is greater or less than unity. It thus appears, that, whatever be the system, all possible positive numbers between 0 and ∞ will embrace for their logarithms all possible numbers between $-\infty$ and $+\infty$. The logarithms of negative numbers, if they admit of expression, must therefore fall in the class of imaginary quantities.

In the equation

$$\cos. x + \sqrt{-1} \sin. x = e^{x\sqrt{-1}} \quad \text{(Eq. 4, Art. 73)},$$

write $x + 2k\pi$ for x, k being any whole number; then

$$\cos. (x + 2k\pi) + \sqrt{-1} \sin. (x + 2k\pi) = e^{(x+2k\pi)\sqrt{-1}}.$$

For $x = 0$, this gives $\quad 1 = e^{2k\pi\sqrt{-1}}$;

for $x = \pi$, this gives $\quad -1 = e^{(2k+1)\pi\sqrt{-1}}$.

Taking the Napierian logarithms of both members of these equations, we have

$$l(1) = 2k\pi\sqrt{-1}, \quad l(-1) = (2k+1)\pi\sqrt{-1}.$$

These are the general expressions for the Napierian logarithms of 1 and -1: and, since k may be any whole number, it follows that both $+1$ and -1 have an infinite number of logarithms; but all of them, except that of $+1$, corresponding to $k=0$, will be imaginary.

From this it may be shown, that any positive or negative number, in whatever system, has an indefinite number of logarithms.

For, first, suppose y to be any positive number, and x its arithmetical logarithm taken in the Napierian system; then

$$y = e^x = e^x \times 1 = e^x \times e^{2k\pi\sqrt{-1}} = e^{x+2k\pi\sqrt{-1}}:$$

$$\therefore \qquad ly = x + 2k\pi\sqrt{-1},$$

which is the general Napierian logarithm of y, and will admit of an unlimited number of values. Denoting the arithmetical logarithm by $l(y)$, we have

$$ly = l(y) + 2k\pi\sqrt{-1} \ldots (m).$$

Again: let Ly denote the general logarithm of y, taken in the system of which a is the base, $L(y)$ denoting the arithmetical logarithm; then, since we pass from Napierian to any other logarithms by multiplying the former by the modulus of the system to which we pass, multiply Eq. m by $\frac{1}{la}$, the modulus of the system characterized by L, which gives

$$ly \times \frac{1}{la} = l(y) \times \frac{1}{la} + \frac{1}{la}\,2k\pi\sqrt{-1};$$

or,

$$Ly = L(y) + \frac{2k\pi\sqrt{-1}}{la} \ldots (n).$$

From Eqs. m, n, we conclude that the arithmetical logarithm of a positive number taken in any system is the value of the general logarithm corresponding to $k=0$.

Now, suppose y to be negative; then $-y = -1 \times y$, and $-1 \times y = e^x \times -1 = e^x \times e^{(2k+1)\pi\sqrt{-1}} = e^{x+(2k+1)\pi\sqrt{-1}}$:

$$\therefore \qquad l(-y) = x + (2k+1)\pi\sqrt{-1} \ldots (p),$$

also
$$L(-y) = \frac{x + (2k+1)\pi\sqrt{-1}}{la} \ldots (q).$$

Eqs. p, q, are the general expressions of the logarithms of a negative number, and show that such a number has an unlimited number of logarithms, all of which are imaginary.

From the equation $l(-1) = (2k+1)\pi\sqrt{-1}$, we get

$$\pi = \frac{l(-1)}{(2k+1)\sqrt{-1}}.$$

This and the preceding remarkable results developed in this section must be interpreted with reference to the symbols and the character of the quantities with which we are dealing. It must be remembered that e and π are the representatives of arithmetical series, and that the formulas have meaning, and can be regarded as expressing true relations, only when the rules for combining imaginary quantities with each other and with real quantities are strictly observed.

SECTION VIII.

DIFFERENTIATION OF EXPLICIT FUNCTIONS OF TWO OR MORE INDEPENDENT VARIABLES, OF FUNCTIONS OF FUNCTIONS, AND OF IMPLICIT FUNCTIONS OF SEVERAL VARIABLES.

80. When several variables are involved in an equation, any one of them may be selected as the function or dependent variable; the others being regarded as independent. If the value of the function is directly expressed in terms of the variables, we have an explicit function of several independent variables; but, when the function and the variables are involved in an unresolved equation, we have an implicit function.

Let $u = F(x, y)$ be an explicit function of the two independent variables, x, y, and give to these variables the increments, Δx, Δy, whereby u receives the increment Δu expressed by the equation

$$\Delta u = F(x + \Delta x, y + \Delta y) - F(x, y) = F(x + \Delta x, y) - F(x, y) \\ + F(x + \Delta x, y + \Delta y) - F(x + \Delta x, y) \ldots (a).$$

The partial derivative, or differential co-efficient, of a function with respect to one of the variables involved in the function, is that which comes from attributing an increment to that variable alone. The partial derivative, or differential co-efficient, of $u = F(x, y)$, taken with respect to x, is denoted by $F'_x(x, y)$, or $\frac{du}{dx}$. In like manner, $F'_y(x, y)$, or $\frac{du}{dy}$, denotes the partial differential co-efficient taken with respect to y; and $F''_{xy}(x, y)$, or

$\frac{d^2u}{dxdy}$, is the partial differential co-efficient taken with respect to x of the partial differential co-efficient taken with respect to y.

Now, if r_1, r_2, r_3, are quantities which vanish with Δx, Δy, then, by Art. 15, we have the following:—

$$F(x+\Delta x, y) - F(x, y) = F'_x(x, y)\Delta x + r_1 \Delta x,$$

$$F(x+\Delta x, y+\Delta y) - F(x+\Delta x, y) = F'_y(x+\Delta x, y)\Delta y + r_2\Delta y,$$

$$F'_y(x+\Delta x, y) - F'_y(x,y) = F''_{xy}(x,y)\Delta x + r_3 \Delta x;$$

from which last we get

$$F'_y(x+\Delta x, y) = F'_y(x, y) + F''_{xy}(x, y)\Delta x + r_3 \Delta x.$$

By substituting these values in Eq. a, it becomes

$$\Delta u = F'_x(x, y)\Delta x + F'_y(x, y)\Delta y + r_1 \Delta x + r_2 \Delta y$$
$$+ F''_{xy}(x, y)\Delta x \Delta y + r_3 \Delta x \Delta y;$$

or,

$$\Delta u = \frac{du}{dx}\Delta x + \frac{du}{dy}\Delta y + r_1 \Delta x + r_2 \Delta y$$
$$+ \frac{d^2u}{dxdy}\Delta x \Delta y + r_3 \Delta x \Delta y \ldots \quad (b).$$

The increment Δu of a function of two independent variables is, therefore, like that of a function of a single variable, composed of two parts; the one, $\frac{du}{dx}\Delta x + \frac{du}{dy}\Delta y$, of the first degree with respect to the increments Δx, Δy, and in which the co-efficients of these increments do not vanish with the increments. The other part is made up of terms which are either of a higher degree than the first with respect to Δx, Δy, or they are terms in which the co-efficients r_1, r_2, r_3, of the first powers of Δx, Δy, vanish with these increments.

From what precedes, we pass by what seems to be a natural extension of our definition, Art. 16, of the differential of a

function of a single variable, to that of a function of two variables. If we write du, dx, dy, for Δu, Δx, Δy, respectively, in Eq. b, neglecting at the same time all the terms in the second member after the second term, we have

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy \quad (c).$$

Here du in the first member denotes the *total* differential of u, and is different from the du in $\frac{du}{dx}, \frac{du}{dy}$. In this, as in former cases of differentiation, $\frac{du}{dx}, \frac{du}{dy}$, are to be regarded as the limits of the ratios of the increments of the variables to the corresponding increments of the function; the distinction being, that now in each of these ratios the increment of the function is partial, and refers to the variable whose increment is the denominator of the ratio. We must treat $\frac{du}{dx}, \frac{du}{dy}$, as wholes, and not as fractions having du for the numerators, and dx, dy, for the denominators. It is true that du, dx, dy, in these differential co-efficients, may be regarded as quantities rather than as the traces of quantities which have vanished, by assigning them such relative values, generally infinitely small, that their ratio shall always be equal to the differential co-efficients. In this case, $\frac{du}{dx}dx$ would reduce to du; but this is the partial differential of u taken with respect to x, and should be written $d_x u$. So likewise $\frac{du}{dy}dy$ should be written $d_y u$. To indicate that $\frac{du}{dx}, \frac{du}{dy}$, are partial differential co-efficients, they are sometimes enclosed in ( ); thus, $\left(\frac{du}{dx}\right), \left(\frac{du}{dy}\right)$.

From Eq. c, we conclude that the total differential of a func-

tion of two independent variables is the sum of the partial differentials taken with respect to each of the variables separately.

81. To find the differential of $u = F(x, y, z)$, a function of the three independent variables x, y, z, denote as before, by $r_1, r_2, r_3 \ldots$, quantities that vanish with Δx, Δy, Δz; then

$$\begin{aligned}\Delta u &= F(x+\Delta x,\ y+\Delta y,\ z+\Delta z) - F(x,y,z)\\ &= F(x+\Delta x, y, z) - F(x,y,z) + F(x+\Delta x,\ y+\Delta y,\ z)\\ &\quad - F(x+\Delta x, y, z) + F(x+\Delta x,\ y+\Delta y,\ z+\Delta z)\\ &\qquad - F(x+\Delta x,\ y+\Delta y,\ z) \ldots \quad (d).\end{aligned}$$

But, Art. 16,

$$F(x+\Delta x, y, z) - F(x, y, z) = F'_x(x, y, z)\,\Delta x + r_1\,\Delta x,$$

$$F(x+\Delta x, y+\Delta y, z) - F(x+\Delta x, y, z)$$
$$= F'_y(x+\Delta x, y, z)\,\Delta y + r_2\,\Delta y.$$

$$\left.\begin{array}{l}F(x+\Delta x, y+\Delta y, z+\Delta z)\\ \quad - F(x+\Delta x, y+\Delta y, z)\end{array}\right\} = F'_z(x+\Delta x, y+\Delta y, z)\,\Delta z + r_3\,\Delta z.$$

Also, from same article,

$$F'_y(x+\Delta x, y, z) - F'_y(x, y, z) = F''_{xy}(x, y, z)\,\Delta x + r_4\,\Delta x;$$

and therefore

$$F'_y(x+\Delta x, y, z) = F'_y(x, y, z) + F''_{xy}(x, y, z)\,\Delta x + r_4\,\Delta x.$$

So, likewise,

$$F_z(x+\Delta x, y+\Delta y, z) = F'_z(x+\Delta x, y, z)$$
$$+ F''_{yz}(x+\Delta x, y, z)\,\Delta y + r_5\,\Delta y,$$

and

$$F'_z(x+\Delta x, y, z) = F'_z(x, y, z) + F''_{xz}(x, y, z)\,\Delta x + r_6\,\Delta x.$$

Making these substitutions in Eq. d, and denoting the coefficients of the terms containing the products of Δx, Δy, Δz, by each other, by m_1, m_2, m_3, we have

$$\Delta u = F'_x(x, y, z)\,\Delta x + F'_y(x, y, z)\,\Delta y + F'_z(x, y, z)\,\Delta z$$
$$+ r_1\,\Delta x + r_2\,\Delta y + r_3\,\Delta z + m_1\,\Delta x\,\Delta y + m_2\,\Delta x\,\Delta z + m_3\,\Delta y\,\Delta z;$$

$$\text{or,}\quad \Delta u = \frac{du}{dx}\Delta x + \frac{du}{dy}\Delta y + \frac{du}{dz}\Delta z + r_1 \Delta x + r_2 \Delta y + r_3 \Delta z$$
$$+ m_1 \Delta x \Delta y + m_2 \Delta x \Delta z + m_3 \Delta y \Delta z.$$

From this, by the same considerations that led us to the expression for the total differential of a function of two independent variables, we conclude that

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz,$$

which may be written

$$du = d_x u + d_y u + d_z u.$$

The course to be followed for a function of four or a greater number of independent variables, and the results at which we should arrive, are obvious. The total differential of a function of any number of independent variables is therefore equal to the sum of the partial differentials of the function taken with respect to each of the variables separately.

82. In Art. 42, a rule was given for the differentiation of a function of an explicit function of a single variable. It is now proposed to treat this subject more generally.

Let $u = F(y, z)$ be a function of the variables y, z, which are themselves functions of a third variable x, and given by the equations $y = \varphi(x)$, $z = \psi(x)$. If x be increased by Δx, u, y, and z will take corresponding increments, which denote by Δu, Δy, Δz; then

$$\Delta u = F(y + \Delta y, z + \Delta z) - F(y, z) = F(y + \Delta y, z)$$
$$- F(y, z) + F(y + \Delta y, z + \Delta z) - F(y + \Delta y, z).$$

Dividing through by Δx, and in the second member multiplying and dividing the first two terms by Δy, and the second two by Δz,

$$\frac{\Delta u}{\Delta x} = \frac{F(y + \Delta y, z) - F(y, z)}{\Delta y}\,\frac{\Delta y}{\Delta x}$$
$$+ \frac{F(y + \Delta y, z + \Delta z) - F(y + \Delta y, z)}{\Delta z}\,\frac{\Delta z}{\Delta x}.$$

Passing to the limit by making $\Delta x = 0$, and remembering that Δy and Δz vanish with Δx, the first member becomes $\frac{du}{dx}$; the first term of the second member becomes $\frac{du}{dy}\frac{dy}{dx}$. To see clearly what the second term of the second member becomes, suppose, first, that Δy vanishes; then this term reduces to

$$\frac{F(y, z + \Delta z) - F(y, z)}{\Delta z} \frac{\Delta z}{\Delta x};$$

and it is evident that, now, the factor $\frac{F(y, z + \Delta z) - F(y, z)}{\Delta z}$ is the ratio of the increment Δz to the corresponding increment of the function: hence, at the limit, this factor becomes $\frac{du}{dz}$, and the second term $\frac{du}{dz}\frac{dz}{dx}$; and therefore we have

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} + \frac{du}{dz}\frac{dz}{dx},$$

and

$$\frac{du}{dx} dx = du = \frac{du}{dy} dy + \frac{du}{dz} dz.$$

In general, if $w = F(y, z, u, v \ldots)$, $y, z, u, v \ldots$ being all functions of the same variable x, we should have

$$\frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx} + \frac{dw}{dz}\frac{dz}{dx} + \frac{dw}{du}\frac{du}{dx} + \ldots \quad (a)$$

$$dw = \frac{dw}{dy} dy + \frac{dw}{dz} dz + \frac{dw}{du} du + \ldots \quad (b).$$

Here $\frac{dw}{dy}, \frac{dw}{dz} \ldots, \frac{dw}{dy} dy, \frac{dw}{dz} dz$, are the partial differential co-efficients and partial differentials of the function w; while $\frac{dw}{dx}$ and dw in the first members of these equations are the total differential co-efficient and total differential

of the function: hence we may ennunciate the following theorem; viz., the differential co-efficient of a function of any number of variables, all of which are functions of the same independent variable, is the algebraic sum of the results obtained by multiplying the partial differential co-efficient of the function taken with respect to each dependent variable by the differential co-efficient of such variable taken with respect to the independent variable. This is the meaning of Eq. *a;* and Eq. *b* admits of a like interpretation.

If in the function, $u = F(y, z)$, we suppose, for a particular case, that y and z in terms of x are given by the equations $y = f(x)$, $z = x$; then $dz = dx$, $\frac{dz}{dx} = 1$, $\frac{du}{dz} = \frac{du}{dx}$; and the second term in the second member of the equation,

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} + \frac{du}{dz}\frac{dz}{dx},$$

would reduce to $\frac{du}{dx}$, which is the partial differential co-efficient of $u = F(y, z) = F(y, x)$ taken with respect to x. This must be in some way distinguished from $\frac{du}{dx}$ in the first member of the equation, which is the total differential co-efficient of the function. This is usually done, in cases where the two kinds of differential co-efficients are likely to be confounded, by enclosing the partial differential co-efficients in a parenthesis. Thus the above equation should then be written

$$\frac{du}{dx} = \left(\frac{du}{dy}\right)\frac{dy}{dx} + \left(\frac{du}{dx}\right).$$

83. It may happen that some of the subordinate functions are themselves functions of the others, and thus complicate the example; but the principle just demonstrated is easily extended to such cases. For example:—

let
$$u = F(y, z, v, x), \quad v = f(y, z, x),$$
$$y = \varphi(x), \qquad z = \psi(x);$$

from which, by making the proper substitutions, u could be made an explicit function of x, and thus the differential co-efficient of u with respect to x be found. But this result may be reached without making these substitutions.

Differentiating each of these equations with respect to x, we have

$$\frac{du}{dx} = \left(\frac{du}{dy}\right)\frac{dy}{dx} + \left(\frac{du}{dz}\right)\frac{dz}{dx} + \left(\frac{du}{dv}\right)\frac{dv}{dx} + \left(\frac{du}{dx}\right),$$

$$\frac{dv}{dx} = \left(\frac{dv}{dy}\right)\frac{dy}{dx} + \left(\frac{dv}{dz}\right)\frac{dz}{dx} + \left(\frac{dv}{dx}\right),$$

$$\frac{dy}{dx} = \varphi'(x), \quad \left(\frac{dz}{dx}\right) = \psi'(x),$$

in which we distinguish partial from total differential co-efficients by enclosing the former in parentheses. By substituting in the first of these differential equations the values of $\frac{dv}{dx}$, $\frac{dy}{dx}$, $\frac{dz}{dx}$, derived from the others, we get, finally,

$$\frac{du}{dx} = \left(\frac{du}{dy}\right)\varphi'(x) + \left(\frac{du}{dz}\right)\psi'(x)$$
$$+ \left(\frac{du}{dv}\right)\left\{\left(\frac{dv}{dy}\right)\varphi'(x) + \left(\frac{dv}{dz}\right)\psi'(x) + \left(\frac{dv}{dx}\right)\right\} + \left(\frac{du}{dx}\right).$$

Ex.
$$u = y^2 + z^3 + z^2 y,$$
$$y = \cos. x, \qquad z = e^x,$$
$$\frac{du}{dy} = 2y + z^2, \quad \frac{du}{dz} = 3z^2 + 2zy,$$
$$\frac{dy}{dx} = -\sin. x, \quad \frac{dz}{dx} = e^x:$$

therefore $\dfrac{du}{dx} = -(2y + z^2)\sin.x + (3z^2 + 2zy)e^x$

$$= -(2\cos.x + e^{2x})\sin.x + (3e^{2x} + 2e^x\cos.x)e^x$$
$$= 3e^{3x} - e^{2x}(\sin.x - 2\cos.x) - \sin.2x;$$

a result identical with that obtained by first substituting in u the values of y and z, and differentiating the explicit function,

$$u = \cos.^2 x + e^{3x} + e^{2x}\cos.x.$$

84. When the relation between the variables is expressed by an unresolved equation, any one of the variables may be assumed as a function of the others regarded as independent. It is often inconvenient, or even impossible, to solve the equation with reference to the variable taken as the function, and thus convert it into an explicit function to which preceding rules for differentiation are applicable; and hence the necessity for investigating special methods for the differentiation of this class of functions.

Consider, first, a function of a single variable, which, in its most general form, may be written $u = F(x, y) = 0$. Either y may be taken as a function of x, or x as a function of y. It generalizes our result to leave the selection of the independent variable undetermined. Let Δx, Δy, be the simultaneous increments of x and y. The increased variables $x + \Delta x$, $y + \Delta y$, are subject to the law of the function $F(x, y) = 0$, and hence must satisfy the equation,

$$F(x + \Delta x, y + \Delta y) = 0:$$

therefore

$$\Delta u = F(x + \Delta x, y + \Delta y) - F(x, y) = 0.$$

Treating $F(x + \Delta x, y + \Delta y) - F(x, y)$ as was done in the case of a function of two independent variables in the last article, we have

$$\Delta u = 0$$
$$= \frac{du}{dx}\Delta x + \frac{du}{dy}\Delta y + r_1\Delta x + r_2\Delta y + \frac{d^2u}{dxdy}\Delta x\Delta y + r_3\Delta x\Delta y\,..\,(a);$$

r_1, r_2, r_3, being quantities that vanish with Δx, Δy.

Now, by whichever of the increments we divide through, and then pass to the limit, by making that increment zero, it is manifest, that since, from the mutual dependence of x and y, Δx and Δy become zero together, all the terms in the second member of the above equation will vanish except the first two.

Dividing through by Δx, and passing to the limit, we have

$$\frac{du}{dx}+\frac{du}{dy}\lim.\frac{\Delta y}{\Delta x}=\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}=0:$$

whence
$$\frac{dy}{dx}=-\frac{\frac{du}{dx}}{\frac{du}{dy}}.$$

Dividing through by Δy, and passing to the limit, we get

$$\frac{du}{dx}\frac{dx}{dy}+\frac{du}{dy}=0:\ \therefore\ \frac{dx}{dy}=-\frac{\frac{du}{dy}}{\frac{du}{dx}}.$$

In Eq. a, writing du, dx, dy, for Δu, Δx, Δy, and omitting all the terms in the second member after the first two, it becomes

$$du=\frac{du}{dx}dx+\frac{du}{dy}dy=0.$$

85. If $u=F(x,y,z\ldots)=0$ be a function of any number of variables, one among them may be taken as a function of all the others regarded as independent. Were the equation solved with reference to the variable selected as dependent, we should then have to deal with an explicit function of several independent variables,—a function which has no total differential co-efficients, such as there are in the case of explicit functions of a single variable; and we are, therefore, concerned only

with the total differentials of the function, and with its partial differential co-efficients of the different orders.

Suppose z to be the dependent variable, and that the value of z, in terms of the other variables, is $z = f(x, y \ldots)$: then

$$u = F\Big(x, y, f(x, y \ldots) \ldots\Big) = 0;$$

and, considered with reference to x alone, u is a function of x, and of a function of a function of x. But, by the law of the function $F(x, y, z \ldots)$, u must be zero for all values of the independent variables: hence its partial differential co-efficients, taken with respect to these variables, must be zero.

Denote by $\left(\frac{du}{dx}\right)$ the partial differential co-efficient of u taken with respect to x, and, through x, with respect to z; and, by $\frac{du}{dx}, \frac{du}{dz}$, the partial differential co-efficient taken with respect to x and z separately: then, by Art. 82,

$$\left(\frac{du}{dx}\right) = \frac{du}{dx} + \frac{du}{dz}\frac{dz}{dx} = 0 \quad (a).$$

Similarly, by adopting a like notation with reference to $y, s, t \ldots$, we have

$$\left(\frac{du}{dy}\right) = \frac{du}{dy} + \frac{du}{dz}\frac{dz}{dy} = 0 \quad (b).$$

$$\left(\frac{du}{ds}\right) = \frac{du}{ds} + \frac{du}{dz}\frac{dz}{ds} = 0 \quad (c).$$

. . . .

Eqs. $a, b, c \ldots$, will give the partial differential co-efficients of z with respect to the variables severally. Thus, from (a), we have

$$\frac{dz}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dz}}, \text{ and, from } (b), \frac{dz}{dy} = -\frac{\frac{du}{dy}}{\frac{du}{dz}} \ldots$$

Multiplying Eqs. $a, b, c \dots$ through by $dx, dy, ds \dots$ respectively, and adding the results, observing that

$$\frac{du}{dz}\frac{dz}{dx}dx + \frac{du}{dz}\frac{dz}{dy}dy + \frac{du}{dz}\frac{dz}{ds}ds + \dots = \frac{du}{dz}dz,$$

we have

$$\frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz + \frac{du}{ds}ds + \dots = 0 \dots (m).$$

From Eq. m, we may find the total differential of any one of the variables regarded as a function of all the others; thus

$$dx = -\frac{\dfrac{du}{dy}dy + \dfrac{du}{dz}dz + \dfrac{du}{ds}ds + \dots}{\dfrac{du}{dx}}.$$

Ex. 1. $$u = a^2y^2 + b^2x^2 - a^2b^2 = 0,$$

$$\frac{du}{dx} = 2b^2x, \quad \frac{du}{dy} = 2a^2y;$$

therefore $$a^2y\frac{dy}{dx} + b^2x = 0 \quad \therefore \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$

From the given equation, we get $y = \frac{b}{a}\sqrt{a^2 - x^2}$, an explicit function of y; and, by differentiation, we obtain directly

$$\frac{dy}{dx} = -\frac{bx}{a\sqrt{a^2 - x^2}} = -\frac{b^2x}{a^2y}.$$

Ex. 2. $$u = y^3 + x^3 - 3axy = 0,$$

$$\frac{du}{dx} = 3x^2 - 3ay, \quad \frac{du}{dy} = 3y^2 - 3ax,$$

$$\frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax} = \frac{ay - x^2}{y^2 - ax};$$

a result that it would be difficult to verify, as was done in Ex. 1.

86. When we have given the two implicit functions,

$$u = F(x, y, z \dots) = 0, \quad v = f(x, y, z \dots) = 0,$$

of the same variables, we should have at the same time $du = 0$, $dv = 0$, from which can be determined the differentials of any two of the variables considered as implicit functions of all the others; and, in general, if the relation between the n variables, $x, y, z \ldots$, is expressed by the m equations, $u = 0, v = 0, w = 0 \ldots$, we should have at the same time the m differential equations,

$$du = 0,\ dv = 0,\ dw = 0 \ldots,$$

and, by means of these, could determine the differentials of m variables regarded as functions of all the others.

If the number of variables exceeds only by 1 the number of equations expressing the relations between them, one of the variables alone can be independent; and we may find the differential co-efficients of all the others regarded as functions of this single variable.

Let us have n equations,

$$\begin{aligned} u_1 &= F_1(x, y, z \ldots t) = 0, \\ u_2 &= F_2(x, y, z \ldots t) = 0, \\ &\cdots\cdots\cdots\cdots \\ u_n &= F_n(x, y, z \ldots t) = 0, \end{aligned}$$

between the $n + 1$ variables $x, y, z \ldots t$.

Differentiating all of these equations with respect to x taken as the independent variable, we have

$$\frac{du_1}{dx} + \frac{du_1}{dy}\frac{dy}{dx} + \frac{du_1}{dz}\frac{dz}{dx} + \cdots + \frac{du_1}{dt}\frac{dt}{dx} = 0,$$

$$\frac{du_2}{dx} + \frac{du_2}{dy}\frac{dy}{dx} + \frac{du_2}{dz}\frac{dz}{dx} + \cdots + \frac{du_2}{dt}\frac{dt}{dx} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{du_n}{dx} + \frac{du_n}{dy}\frac{dy}{dx} + \frac{du_n}{dz}\frac{dz}{dx} + \cdots + \frac{du_n}{dt}\frac{dt}{dx} = 0.$$

There are n of these *differential* equations involving the n

required quantities, $\frac{dy}{dx}, \frac{dz}{dx} \ldots \frac{dt}{dx}$, which may therefore be determined.

87. When the variables enter the function in certain combinations, the results of differentiation take special forms, and peculiar relations exist between the partial differential co-efficients, depending on the manner in which the variables are combined. We shall first consider the case of homogeneous functions. A function is said to be homogeneous when all the terms entering under the functional symbol are of the same degree with reference to the variables. Thus

$$F(x, y, z) = ax^2 + by^2 + cz^2 + 2eyz$$

is a homogeneous function of 2 dimensions, and

$$F(x, y) = \frac{x}{y}$$

is a homogeneous function of 0 dimensions. A property of such function is, that, if all the variables are multiplied by the same quantity, we obtain for the result the original function multiplied by this quantity raised to a power whose exponent is the number denoting the dimensions of the function. Therefore, if $F(x, y, z \ldots)$ is a homogeneous function of a dimensions, and t denotes a new and independent variable, we have

$$F(tx, ty, tz \ldots) = t^a F(x, y, z \ldots).$$

Put $$tx = u,\ ty = v,\ tz = w \ldots, \text{ then}$$

$$F(u, v, w \ldots) = t^a F(x, y, z \ldots);$$

and differentiate both members of this equation with respect to t: the result is,

$$\frac{dF}{du}\frac{du}{dt} + \frac{dF}{dv}\frac{dv}{dt} + \frac{dF}{dw}\frac{dw}{dt} \ldots = at^{a-1} F(x, y, z \ldots).$$

But $$\frac{du}{dt} = x, \ \frac{dv}{dt} = y, \ \frac{dw}{dt} = z \ldots :$$

therefore

$$x\frac{dF}{du} + y\frac{dF}{dv} + z\frac{dF}{dw} \ldots = at^{a-1}F(x, y, z \ldots).$$

Now, since t is entirely arbitrary, make $t = 1$; then

$$u = x, \ v = y, \ w = z \ldots, \text{ and } \frac{dF}{du} = \frac{dF}{dx}, \ \frac{dF}{dv} = \frac{dF}{dy} \ldots :$$

whence we have

$$x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz} \ldots = aF(x, y, z \ldots).$$

The first member of this equation is the sum of the products obtained by multiplying the partial differential co-efficients of the function, each by the variable to which it relates; and the second member is the primitive function multiplied by the number denoting the degree of the function.

If the function is of 0 degree,

$$x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz} + \ldots = 0.$$

Ex. 1. $F(x, y, z) = ax^2 + by^2 + cz^2 + 2eyz + 2fzx + 2gxy$,

$$\frac{dF}{dx} = 2ax + 2fz + 2gy, \quad \frac{dF}{dy} = 2by + 2cz + 2gx,$$

$$\frac{dF}{dz} = 2cz + 2ey + 2fx,$$

and $a = 2$: therefore

$$\left.\begin{array}{l}(2ax + 2fz + 2gy)x \\ + (2by + 2ez + 2gx)y \\ + (2cz + 2ey + 2fx)z\end{array}\right\} = 2(ax^2 + by^2 + cz^2 + 2eyz + 2fzx + 2gxy),$$

an identical equation.

Ex. 2. $\qquad F(x, y) = \frac{x}{y}, \quad \frac{dF}{dx} = \frac{1}{y}, \quad \frac{dF}{dy} = -\frac{x}{y^2},$

$a = 0$: therefore

$$x\frac{dF}{dx} + y\frac{dF}{dy} = \frac{x}{y} - \frac{xy}{y^2} = \frac{x}{y} - \frac{x}{y} = 0.$$

88. Let us next take the case of the function of the algebraic sum of several variables, $x, y, z \ldots$ If the function be $u = F(x \pm y \pm z \pm \cdots)$, and we put $x \pm y \pm z \pm \cdots = t$, it becomes $u = F(t)$.

Now, if the original function be differentiated with respect to $x, y, z \ldots$ separately, we shall have, by reason of the equation $u = F(t)$,

$$\frac{dF}{dx} = \frac{dF}{dt}\frac{dt}{dx}, \quad \frac{dF}{dy} = \frac{dF}{dt}\frac{dt}{dy}, \quad \frac{dF}{dz} = \frac{dF}{dt}\frac{dt}{dz} \cdots$$

But the equation $x \pm y \pm z \pm \cdots = t$ gives

$$\frac{dt}{dx} = 1 = \pm\frac{dt}{dy} = \pm\frac{dt}{dz} = \pm \cdots :$$

therefore

$$\frac{dF}{dx} = \pm\frac{dF}{dy} = \pm\frac{dF}{dz} = \pm \cdots;$$

that is, the partial differential co-efficients of the function are numerically equal.

Ex. 1. $\qquad u = (x + y)^n, \qquad \frac{du}{dx} = \frac{du}{dy} = n(x + y)^{n-1}.$

Ex. 2. $\qquad u = (x - y)^n, \qquad \frac{du}{dx} = -\frac{du}{dy} = n(x - y)^{n-1}.$

Ex. 3. $\qquad u = l\sqrt{x + y}, \qquad \frac{du}{dx} = \frac{du}{dy} = \frac{1}{2(x + y)}.$

Ex. 4. $\qquad u = l\sqrt{x - y}, \qquad \frac{du}{dx} = -\frac{du}{dy} = \frac{1}{2(x - y)}.$

SECTION IX.

SUCCESSIVE DIFFERENTIATION OF FUNCTIONS OF TWO OR MORE INDEPENDENT VARIABLES, AND OF IMPLICIT FUNCTIONS.

89. In Sect. IV., rules were investigated for the successive differentiation of explicit functions of a single variable. We now pass to the successive differentiation of functions of many variables, all of which, at first, will be supposed independent of each other.

By Art. 81, the total differential of a function of several variables is the algebraic sum of its partial differentials; and it is evident that each partial differential co-efficient is, in general, a function of these variables, which may be again differentiated with respect to a part of the variables, or with respect to the whole of them. These operations give rise to what are called partial and total differentials, and differential co-efficients of the different orders.

90. If $u = F(x, y, z \ldots)$ be a function of the independent variables $x, y, z \ldots$, then du, d^2u, $d^3u \ldots d^nu \ldots$, standing by themselves, will denote the first, second, third ... n^{th} total differentials of the function. $\dfrac{du}{dx}$ is the first partial differential co-efficient of u taken with respect to x; and $\dfrac{du}{dx}dx$, or $d_x u$, is the corresponding partial differential. $\dfrac{d}{dx}\left(\dfrac{du}{dx}\right) = \dfrac{d^2u}{dx^2}$ is the second partial differential co-efficient with respect to x;

and the differential corresponding to it is $\frac{d^2u}{dx^2}dx^2$, or $d^2_x u$.

$\frac{d}{dy}\left(\frac{du}{dx}\right)=\frac{d^2u}{dydx}$ is the second partial differential co-efficient, taken, first with respect to x, and then with respect to y; and the differential answering to it is $\frac{d^2u}{dydx}dydx$, or $d_y d_x u$.

$\frac{d}{dy}\left(\frac{d^2u}{dx^2}\right)=\frac{d^3u}{dydx^2}$ is the third partial differential co-efficient of the function obtained by differentiating twice with respect to x, and then once with respect to y; and to this we have the corresponding differential $\frac{d^3u}{dydx^2}dydx^2$, or $d^2_x d_y u$, which may also be denoted by $d^3_{x^2y}u$. In like manner, the notations $\frac{d^4u}{dx^2dydz}$, $\frac{d^4u}{dx^2dydz}dx^2dydz$, $d^2_x d_y d_z u$, $d^4_{x^2yz}u$, would indicate four differentiations: one with respect to z, one with respect to y, and two with respect to x. From what precedes, the signification of the notations $\frac{d^{m+n+p\cdots}u}{dx^m dy^n dz^p\ldots}$, $\frac{d^{m+n+p\cdots}u}{dx^m dy^n dz^p\ldots}dx^m dy^n dz^p$, $d^m_x d^n_y d^p_z\ldots u$, $d^{m+n+p\cdots}_{x^m y^n z^p\ldots}u$, will be readily understood.

The remarks in Art. 81, in reference to partial differential co-efficients of the first order, are equally applicable to those of the higher orders. Keeping in view the principles there laid down, there will be no risk of confounding any order of partial differential of the function with the total differential of the same order. Thus, in $\frac{d^2u}{dxdy}$, the d^2u is always associated with $dxdy$ written below it; and in this way the construction of the expression indicates both the character of the differential co-efficient, and the variables with reference to which it is taken.

It is often convenient to attach to the symbol of the func-

tion the characters by which are denoted the order of differentiation, and the variables involved in the operation. Thus, $F'_x(x, y, z \ldots)$, $F''_{xy}(x, y, z \ldots)$, $F'''_{x^2y}(x, y, z \ldots)$, have respectively the same significance as $\dfrac{du}{dx}$, $\dfrac{d^2u}{dxdy}$, $\dfrac{d^3u}{dx^2dy}$, as above explained.

91. Before proceeding farther, we must prove, that, in whatever order in respect to the variables the differentiation of a function of many independent variables is effected, the result is always the same: that is, if $u = F(x, y, z \ldots)$ is to be differentiated m times with respect to x, and n times with respect to y, the result is the same, whether we perform the m x-differentiations, and then the n y-differentiations, or reverse the order of differentiation in respect to x and y; or perform first a part of the m x-differentiations, then a part of the n y-differentiations; and so on until the whole of the m and n differentiations are effected.

This principle may be demonstrated as follows: Take the function $u = F(x, y, z \ldots)$ of the independent variables $x, y, z \ldots$ Suppose, in the first instance, x to be variable, and all the other variables constant, give x the increment h, and develop by the formula of Art. 61; then, in the result, suppose y to be variable, and all the other variables, including x, to be constant, give y the increment k, and develop the terms by the same formula. The final result will be the same as that we should have reached by giving the increments to x and y simultaneously.

Changing x into $x + h$, then, Art. 61,

$$F(x+h, y, z \ldots) = F(x, y, z \ldots) + hF'_x(x, y, z \ldots) + \frac{h^2}{2} F''_{x^2}(x + \theta_1 h, y, z \ldots) \ldots \quad (1);$$

in which $F''_{x^2}(x+\theta_1 h, y, z \ldots)$ is a function $x, h, y, z \ldots$, which remains finite when $h=0$.

If in (1) we change y into $y+k$, the first member becomes

$$F(x+h, y+k, z \ldots);$$

and the terms in the second member become respectively

$$F(x, y+k, z \ldots) = F(x, y, z \ldots) + kF'_y(x, y, z \ldots) + \frac{k^2}{2} F''_{y^2}(x, y+\theta_2 k, z \ldots),$$

$$hF'_x(x, y+k, z \ldots) = hF'_x(x, y, z \ldots) + hkF''_{xy}(x, y, z \ldots) + \frac{hk^2}{2} F'''_{xy^2}(x, y+\theta_3 k, z \ldots)$$

$$\frac{h^2}{2} F''_{x^2}(x+\theta_1 h, y+k, z \ldots) = \frac{h^2}{2} F''_{x^2}(x+\theta_1 h, y, z \ldots) + \frac{h^2 k}{2} F'''_{x^2 y}(x+\theta_1 h, y+\theta_4 k, z \ldots).$$

Making these substitutions in Eq. 1, we find

$$F(x+h, y+k, z \ldots) = \left\{ \begin{aligned} & F(x, y, z \ldots) + hF'_x(x, y, z \ldots) \\ & \qquad + kF'_y(x, y, z \ldots) \\ & \qquad + hkF''_{xy}(x, y, z \ldots) \\ & + \frac{h^2}{2} F''_{x^2}(x+\theta_1 h, y, z \ldots) \\ & + \frac{k^2}{2} F''_{y^2}(x, y+\theta_2 k, z \ldots) \\ & + \frac{hk^2}{2} F'''_{xy^2}(x, y+\theta_3 k, z \ldots) \\ & + \frac{h^2 k}{2} F'''_{x^2 y}(x+\theta_1 h, y+\theta_4 k, z \ldots). \end{aligned} \right.$$

If we begin by giving y its increment, we shall have the equation

$$F(x, y+k, z \ldots) = F(x, y, z \ldots) + kF'_y(x, y, z \ldots) + \frac{k^2}{2} F''_{y^2}(x, y+\theta_2 k, z \ldots);$$

and in this, giving to x its increment h, and developing the terms as was done above, we have

$$F(x+h, y+k, z\ldots) = \begin{cases} F(x, y, z\ldots) + hF'_x(x, y, z\ldots) \\ \qquad\qquad + kF'_y(x, y, z\ldots) \\ \qquad\qquad\quad khF''_{yx}(x, y, z\ldots) \\ + \frac{h^2}{2} F''_{x^2}(x+\theta_1 h, y, z\ldots) \\ + \frac{k^2}{2} F''_{y^2}(x, y+\theta_2 k, z\ldots) \\ + \frac{kh^2}{2} F'''_{yx^2}(x+\theta' h, y, z\ldots) \\ + \frac{k^2 h}{2} F'''_{y^2x}(x+\theta'' h, y+\theta_2 k, z\ldots). \end{cases}$$

It is to be observed, that, in the several preceding equations, the factors $F''_{x^2}(x+\theta_1 h, y, z\ldots)$, $F'''_{xy^2}(x, y+\theta_3 k, z\ldots)$, &c., of terms in the second members, remain finite when h and k, separately or together, become zero.

Equating these two values of $F(x+h, y+k, z\ldots)$, suppressing terms common to the two members of the resulting equation, and dividing through by hk, we have

$$\left. \begin{array}{l} F''_{xy}(x, y, z\ldots) + \frac{k}{2} F'''_{xy^2}(x, y+\theta_3 k, z\ldots) \\ \qquad\qquad + \frac{h}{2} F'''_{x^2y}(x+\theta_1 h, y+\theta_4 k, z) \end{array} \right\}$$

$$= \begin{cases} F''_{yx}(x, y, z) + \frac{h}{2} F'''_{yx^2}(x+\theta' h, y, z\ldots) \\ \qquad\qquad + \frac{k}{2} F'''_{y^2x}(x+\theta'' h, y+\theta_2 k, z\ldots). \end{cases}$$

This equation must be true, whatever the values of h and k. Make $h=0$, $k=0$; then

$$F''_{xy}(x, y, z\ldots) = F''_{yx}(x, y, z) \quad (2).$$

The first member of this equation is the second partial differential co-efficient of the function obtained by differentiating, first with respect to x, and then with respect to y; the second member is the second partial differential co-efficient

which comes from differentiating, first with respect to y, and then with respect to x. It is therefore immaterial in what order the differentiations are performed.

This theorem being demonstrated for derivatives and differentials of the second order, it can easily be extended to those of any order. Suppose we start with $\frac{du}{dz}$. Whether this be differentiated, first with respect to x, and then with respect to y, or we invert the order of differentiation, the result is the same by what has been proved. So that

$$\frac{d^3u}{dxdydz} = \frac{d^3u}{dydxdz}.$$

But the order of differentiation with respect to z, and either x or y, may also be inverted; and therefore

$$\frac{d^3u}{dxdydz} = \frac{d^3u}{dydxdz} = \frac{d^3u}{dxdzdy};$$

and generally, for the function $u = F(x, y, z \ldots)$,

$$\frac{d^{m+n+p}u}{dx^m dy^n dz^p} = \frac{d^{n+m+p}u}{dy^n dx^m dz^p} = \frac{d^{m+p+n}u}{dx^m dz^p dy^n}.$$

Ex. 1. $$u = \frac{x^2 - y^2}{x^2 + y^2},$$

$$\frac{du}{dx} = \frac{4xy^2}{(x^2 + y^2)^2}, \qquad \frac{du}{dy} = -\frac{4x^2y}{(x^2 + y^2)^2},$$

$$\frac{d^2u}{dydx} = \frac{8xy(x^2 - y^2)}{(x^2 + y^2)^3}, \quad \frac{d^2u}{dxdy} = \frac{8xy(x^2 - y^2)}{(x^2 + y^2)^3}.$$

Ex. 2. $$u = \tan.^{-1}\frac{x}{y},$$

$$\frac{du}{dx} = \frac{y}{x^2 + y^2}, \qquad \frac{du}{dy} = -\frac{x}{x^2 + y^2},$$

$$\frac{d^2u}{dydx} = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad \frac{d^2u}{dxdy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

92. The differential of an independent variable is a constant, and therefore the differential of the differential of such a variable is zero. Now, since the differential co-efficient of a function of the independent variables, x, y, z, with respect to one of these variables, is a new function of x, y, z, and the corresponding differential is obtained by multiplying the differential co-efficient by the differential of the variable to which it relates, it follows that, in subjecting such differential of the function to further differentiation, we may set aside the differential of the variable as a constant factor, and operate on the differential co-efficient alone; restoring in our final result the constant factors set aside: thus, if $u = F(x, y, z)$, in which x, y, and z are independent, then

$$d_x u = F'_x(x, y, z)dx = \frac{du}{dx}dx,$$

$$d_y d_x u = dx d_y F'_x(x, y, z) = dx F''_{yx}(x, y, z)dy$$

$$= F''_{yx}(x, y, z)dxdy = \frac{d^2u}{dxdy}dxdy,$$

$$d_z d_y d_x u = dxdyd_z F''_{yx}(x, y, z) = dxdyF'''_{zyx}(x, y, z)dz$$

$$= F'''_{zyx}(x, y, z)dxdydz = \frac{d^3u}{dxdydz}dxdydz,$$

and, generally,

$$d^p_z d^n_y d^m_x u = F^{(m+n+p)}_{z^p y^n x^m}(x, y, z)dx^m dy^n dz^p$$

$$= \frac{d^{m+n+p}u}{dx^m dy^n dz^p}dx^m dy^n dz^p.$$

93. By Art. 81, the first total differential of the function $u = F(x, y, z)$, of the variables x, y, z, is

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz \quad \ldots (1).$$

Taking the total differential of each of the partial differential co-efficients $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, we have

$$d\left(\frac{du}{dx}\right) = \frac{d^2u}{dx^2}dx + \frac{d^2u}{dxdy}dy + \frac{d^2u}{dxdz}dz,$$

$$d\left(\frac{du}{dy}\right) = \frac{d^2u}{dydx}dx + \frac{d^2u}{dy^2}dy + \frac{d^2u}{dydz}dz,$$

$$d\left(\frac{du}{dz}\right) = \frac{d^2u}{dzdx}dx + \frac{d^2u}{dzdy}dy + \frac{d^2u}{dz^2}dz;$$

and therefore

$$d^2u = \frac{d^2u}{dx^2}dx^2 + \frac{d^2u}{dy^2}dy^2 + \frac{d^2u}{dz^2}dz^2 + 2\frac{d^2u}{dxdy}dxdy$$
$$+ 2\frac{d^2u}{dxdz}dxdz + 2\frac{d^2u}{dydz}dydz \quad (2).$$

Proceeding with (2) in the same manner that we did with (1), we should get the third total differential of the function; and so on.

For the function $u = F(x, y)$ of the two independent variables x and y, the successive total differentials will be

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy,$$

$$d^2u = \frac{d^2u}{dx^2}dx^2 + 2\frac{d^2u}{dxdy}dxdy + \frac{d^2u}{dz^2}dz^2,$$

$$d^3u = \frac{d^3u}{dx^3}dx^3 + 3\frac{d^3u}{dx^2dy}dx^2dy + 3\frac{d^3u}{dxdy^2}dxdy^2$$
$$+ \frac{d^3u}{dy^3}dy^3.$$

. . . .

$$d^nu = \frac{d^nu}{dx^n}dx^n + n\frac{d^nu}{dx^{n-1}dy}dx^{n-1}dy$$
$$+ \frac{n(n-1)}{1.2}\frac{d^nu}{dx^{n-2}dy^2}dx^{n-2}dy^2 + \cdots$$
$$+ \frac{n(n-1)\cdots\big(n-(n-2)\big)}{1.2\cdots(n-1)}\frac{d^nu}{dxdy^{n-1}}dxdy^{n-1}$$
$$+ \frac{d^nu}{dy^n}dy^n;$$

the law of the co-efficients being the same as that in the development of $(1+x)^n$.

Ex. 1.
$$u = xyz,$$
$$du = yzdx + xzdy + xydz,$$
$$d^2u = 2(xdydz + ydxdz + zdxdy),$$
$$d^3u = 6dxdydz.$$

Ex. 2. $u = (x^2+y^2)^{\frac{1}{2}}$,

$$\frac{du}{dx} = \frac{x}{(x^2+y^2)^{\frac{1}{2}}},\ \frac{d^2u}{dx^2} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}},\ \frac{d^3u}{dx^3} = -\frac{3xy^2}{(x^2+y^2)^{\frac{5}{2}}},$$
$$\frac{du}{dy} = \frac{y}{(x^2+y^2)^{\frac{1}{2}}},\ \frac{d^2u}{dy^2} = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}},\ \frac{d^3u}{dy^3} = -\frac{3x^2y}{(x^2+y^2)^{\frac{5}{2}}},$$
$$\frac{d^2u}{dxdy} = -\frac{xy}{(x^2+y^2)^{\frac{3}{2}}},\ \frac{d^3u}{dx^2dy} = \frac{y(2x^2-y^2)}{(x^2+y^2)^{\frac{5}{2}}}$$
$$\frac{d^3u}{dxdy^2} = \frac{x(2y^2-x^2)}{(x^2+y^2)^{\frac{5}{2}}}:$$

$$\therefore\quad d^3u = \Big(-3xy^2dx^3 + 3y(2x^2-y^2)dx^2dy + 3x(2y^2-x^2)dxdy^2 - 3yx^2dy^3\Big)\frac{1}{(x^2+y^2)^{\frac{5}{2}}}.$$

Ex. 3. $u = e^{ax+by}$,

$$\frac{du}{dx} = ae^{ax+by},\ \frac{d^2u}{dx^2} = a^2e^{ax+by},$$
$$\frac{du}{dy} = be^{ax+by},\ \frac{d^2u}{dy^2} = b^2e^{ax+by},$$
$$\frac{d^2u}{dxdy} = abe^{ax+by}:$$
$$\therefore\quad d^2u = (a^2dx^2 + 2abdxdy + b^2dy^2)e^{ax+by}$$
$$= (adx + bdy)^2e^{ax+by}.$$

94. If, in the function $s = F(u, v, w)$, u, v, and w are functions of the independent variables x, y, and z, we have a

case of a function of functions of independent variables; and the first total differential of s is

$$ds = \frac{ds}{dx}dx + \frac{ds}{dy}dy + \frac{ds}{dz}dz \quad (1).$$

But since u, v, and w are all functions of x, y, and z, the partial differential of s, regarded as a function of x, is (Art. 82),

$$\frac{ds}{dx}dx = \frac{ds}{du}\frac{du}{dx}dx + \frac{ds}{dv}\frac{dv}{dx}dx + \frac{ds}{dw}\frac{dw}{dx}dx:$$

so also $$\frac{ds}{dy}dy = \frac{ds}{du}\frac{du}{dy}dy + \frac{ds}{dv}\frac{dv}{dy}dy + \frac{ds}{dw}\frac{dw}{dy}dy;$$

and $$\frac{ds}{dz}dz = \frac{ds}{du}\frac{du}{dz}dz + \frac{ds}{dv}\frac{dv}{dz}dz + \frac{ds}{dw}\frac{dw}{dz}dz.$$

The total differential of u is

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz;$$

and, for the total differentials of v and w, we have like expressions: therefore, by substituting these values of $\frac{ds}{dx}dx$, $\frac{ds}{dy}dy$, $\frac{ds}{dz}dz$ in (1), and uniting terms, we have

$$ds = \frac{ds}{du}du + \frac{ds}{dv}dv + \frac{ds}{dw}dw.$$

A second differentiation would give

$$d^2s = \frac{d^2s}{du^2}du^2 + \frac{d^2s}{dv^2}dv^2 + \frac{d^2s}{dw^2}dw^2 + 2\frac{d^2s}{dudv}dudv$$
$$+ 2\frac{d^2s}{dudw}dudw + 2\frac{d^2s}{dvdw}dvdw + \frac{ds}{du}d^2u$$
$$+ \frac{ds}{dv}d^2v + \frac{ds}{dw}d^2w;$$

and from this we pass to d^3s, and so on.

The general rule is, then, to differentiate as if u, v, w, were independent variables, and substitute in the results the values

of du, dv, dw; $d^2u, d^2v, d^2w \ldots$, derived from the equations giving u, v, w, in terms of the independent variables x, y, z.

If $u = ax + by + cz + d,\ v = a'x + b'y + c'z + d',$

$$w = a''x + b''y + c''z + d'',$$

are the expressions for u, v, w, in terms of x, y, z, these functions of the first degree, with respect to the independent variables, are said to be linear. In this case, we should have

$$d^2u = 0,\ d^2v = 0,\ d^2w = 0,\ d^3u = 0 \cdots;$$

and the successive differentials of $s = F(u, v, w)$ would then have the form of the successive differentials of a function of three independent variables: thus

$$d^2s = \frac{d^2s}{du^2}du^2 + \frac{d^2s}{dv^2}dv^2 + \frac{d^2s}{dw^2}dw^2 + 2\frac{d^2s}{dudv}dudv$$
$$+ 2\frac{d^2s}{dudw}dudw + 2\frac{d^2s}{dvdw}dvdw.$$

Ex. 1. $s = F(u, v),\ u = ax + by + c,\ v = a'x + b'y + c',$

$$d^ns = \frac{d^ns}{du^n}du^n + n\frac{d^ns}{du^{n-1}dv}du^{n-1}dv + \cdots \frac{d^ns}{dv^n}dv^n.$$

Ex. 2. $s = F(u)F(v),\ u = ax + by + c,\ v = a'x + b'y + c',$

$$ds = F'(u)F(v)du + F(u)F'(v)dv,$$
$$d^2s = F''(u)F(v)du^2 + 2F'(u)F'(v)dudv + F(u)F''(v)dv^2$$

. . . .

$$d^ns = F^{(n)}(u)F(v)du^n + nF^{(n-1)}(u)F'(v)du^{n-1}dv + \cdots$$
$$+ nF'(u)F^{(n-1)}(v)dudv^{n-1} + F(u)F^n(v)dv^n \ldots$$

95. If the function $s = F(x, y, z)$ of the independent variables, x, y, z, is homogeneous, and of a dimensions, then, by Art. 87,

$$aF(x, y, z) = x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz}.$$

It may be shown that similar relations exist between the function and its differential co-efficients of the higher orders.

Since the function is homogeneous, if we change x, y, and z into tx, ty, tz, and make $u = tx$, $v = ty$, $w = tz$, we have

$$F(u, v, w) = t^a F(x, y, z).$$

Differentiating this twice with respect to t, observing in the second differentiation that $\frac{d}{dt}\left(\frac{du}{dt}\right)$, $\frac{d}{dt}\left(\frac{dv}{dt}\right)$, $\frac{d}{dt}\left(\frac{dw}{dt}\right)$, are each zero, we find,

$$\frac{dF}{du}\frac{du}{dt} + \frac{dF}{dv}\frac{dv}{dt} + \frac{dF}{dw}\frac{dw}{dt} = at^{a-1}F(x, y, z).$$

$$\left.\begin{array}{r} \frac{d^2F}{du^2}\frac{du^2}{dt^2} + \frac{d^2F}{dv^2}\frac{dv^2}{dt^2} + \frac{d^2F}{dw^2}\frac{dw^2}{dt^2} \\ + 2\frac{d^2F}{dudv}\frac{du}{dt}\frac{dv}{dt} + 2\frac{d^2F}{dudw}\frac{du}{dt}\frac{dw}{dt} \\ + 2\frac{d^2F}{dvdw}\frac{dv}{dt}\frac{dw}{dt} \end{array}\right\} = a(a-1)t^{a-2}F(x, y, z).$$

But $\frac{du}{dt} = x$, $\frac{dv}{dt} = y$, $\frac{dw}{dt} = z$; and, if $t = 1$, the second partial differential co-efficients of the function with respect to u, v, w, become the second partial differential co-efficients with respect to x, y, z, respectively; and hence the last of the above equations becomes

$$\left.\begin{array}{r} x^2\frac{d^2F}{dx^2} + y^2\frac{d^2F}{dy^2} + z^2\frac{d^2F}{dz^2} \\ + 2xy\frac{d^2F}{dxdy} + 2xz\frac{d^2F}{dxdz} + 2yz\frac{d^2F}{dydz} \end{array}\right\} = a(a-1)t^{a-2}F(x, y, z).$$

By a third differentiation, we should get

$$\left.\begin{array}{l} x^3\frac{d^3F}{dx^3} + y^3\frac{d^3F}{dy^3} + z^3\frac{d^3F}{dz^3} \\ + 3x^2y\frac{d^3F}{dx^2dy} + 3xy^2\frac{d^3F}{dxdy^2} + \dots \end{array}\right\} = a(a-1)(a-2)t^{a-3}F(x, y, z).$$

Example. $F(x,y,z)=ax^2+by^2+cz^2+2exy+2fxz+2gyz,$

$$\frac{d^2F}{dx^2}=2a,\quad \frac{d^2F}{dy^2}=2b,\quad \frac{d^2F}{dz^2}=2c,$$

$$\frac{d^2F}{dxdy}=2e,\ \frac{d^2F}{dxdz}=2f,\ \frac{d^2F}{dydz}=2g:$$

$$x^2\frac{d^2F}{dx^2}+y^2\frac{d^2F}{dy^2}+z^2\frac{d^2F}{dz^2}+2xy\frac{d^2F}{dxdy}$$

$$+2xz\frac{d^2F}{dxdz}+2yz\frac{d^2F}{dydz}=$$

$$2(ax^2+by^2+cz^2+2exy+2fxz+2gyz)=2F(x,y,z).$$

96. To express the successive differential co-efficients of implicit functions, take the function $u=F(x,y)=0$, in which y is implicitly a function of x; then, by Art. 84,

$$\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}=0 \quad (1).$$

The first member of (1) is another function of x and y, which denote by v; whence $v=0$. Differentiating $v=0$ as we did $u=0$, we have

$$\frac{dv}{dx}+\frac{dv}{dy}\frac{dy}{dx}=0 \quad (2);$$

but $$\frac{dv}{dx}=\frac{d^2u}{dx^2}+\frac{d^2u}{dxdy}\frac{dy}{dx}+\frac{du}{dy}\frac{d^2y}{dx^2},$$

and $$\frac{dv}{dy}=\frac{d^2u}{dxdy}+\frac{d^2u}{dy^2}\frac{dy}{dx}.$$

These values of $\frac{dv}{dx}$, $\frac{dv}{dy}$, substituted in (2), give

$$\frac{d^2u}{dx^2}+2\frac{d^2u}{dxdy}\frac{dy}{dx}+\frac{d^2u}{dy^2}\left(\frac{dy}{dx}\right)^2+\frac{du}{dy}\frac{d^2y}{dx^2}=0 \quad (3).$$

From (1) and (3), we find the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$. Eqs. 1 and 3 are called differential or derived equations of the first and second orders respectively; and, with reference to them, $u=F(x,y)=0$ is the primitive equation.

The above process is somewhat simplified by putting $\frac{dy}{dx}=p$; then

$$v=\frac{du}{dx}+\frac{du}{dy}p=0 \quad (1').$$

Hence $$\frac{dv}{dx}=\frac{d^2u}{dx^2}+\frac{d^2u}{dxdy}p+\frac{du}{dy}\left(\frac{dp}{dx}\right),$$

and $$\frac{dv}{dy}=\frac{d^2u}{dxdy}+\frac{d^2u}{dy^2}p+\frac{du}{dy}\left(\frac{dp}{dy}\right).$$

These values in (2) give

$$\left\{\frac{d^2u}{dy^2}p+\frac{du}{dy}\left(\frac{dp}{dy}\right)+\frac{d^2u}{dxdy}\right\}p+\frac{d^2u}{dxdy}p+\frac{du}{dy}\left(\frac{dp}{dx}\right)+\frac{d^2u}{dx^2}=0,$$

or $$\frac{d^2u}{dx^2}+2\frac{d^2u}{dxdy}p+\frac{d^2u}{dy^2}p^2+\frac{du}{dy}\left\{\left(\frac{dp}{dy}\right)p+\left(\frac{dp}{dx}\right)\right\}=0.$$

But $\left(\frac{dp}{dy}\right)p+\left(\frac{dp}{dx}\right)=\frac{dp}{dx}=\frac{d^2y}{dx^2}$ (Art. 82); and hence

$$\frac{d^2u}{dx^2}+2\frac{d^2u}{dxdy}p+\frac{d^2u}{dy^2}p^2+\frac{du}{dy}\frac{dp}{dx}=0\ldots \quad (3'),$$

or $$\frac{d^2u}{dx^2}+2\frac{d^2u}{dxdy}\frac{dy}{dx}+\frac{d^2u}{dy^2}\left(\frac{dy}{dx}\right)^2+\frac{du}{dy}\frac{d^2y}{dx^2}=0.$$

We call attention to the notations $\left(\frac{dp}{dx}\right)$, $\left(\frac{dp}{dy}\right)$, $\frac{dp}{dx}$, by remarking that p is generally a function of x and y; and that $\left(\frac{dp}{dx}\right)$, $\left(\frac{dp}{dy}\right)$, are the partial differential co-efficients of this function, the first with respect to x, and the second with respect to y: whereas $\frac{dp}{dx}$ is the differential co-efficient of p with respect to x, p being considered, as it is, a function both of x and of a function of a function of x. Thus suppose, that, by solving the primitive equation $F(x,y)=0$, we find $y=f(x)$, then

$$p = f'(x),\ \frac{dp}{dx} = f''(x).$$

Suppose also, that, without solving the primitive equation, we find $p = \varphi(x, y) = \varphi\Big(x, f(x)\Big)$; then

$$\left(\frac{dp}{dx}\right) = \varphi'_x(x, y),\ \left(\frac{dp}{dy}\right) = \varphi'_y(x, y) \dots \quad (a).$$

But, by Art. 82,

$$\frac{dp}{dx} = \varphi'_x(x, y) + \varphi'_y(x, y)\frac{dy}{dx} = f''(x) \dots \quad (b).$$

This points out the necessity of distinguishing, in certain cases, partial differential co-efficients, such as those of p in Eq. a, by the parenthesis, or some other mark, that, in the course of an investigation, they may not be mistaken for others, as $\frac{dp}{dx}$ in Eq. b, of the same form, but having a different significance.

The value of $\frac{d^2y}{dx^2}$ deduced from Eqs. 1 and 3, or from 1′ and 3′, is

$$\frac{d^2y}{dx^2} = -\frac{\frac{d^2u}{dx^2}\left(\frac{du}{dy}\right)^2 - 2\frac{d^2u}{dxdy}\,\frac{du}{dx}\,\frac{du}{dy} + \frac{d^2u}{dy^2}\left(\frac{du}{dx}\right)^2}{\left(\frac{du}{dy}\right)^3}.$$

The expressions for the higher orders of differential co-efficients of implicit functions are so complicated, and so little used, that it is unnecessary to proceed farther with this division of the subject; but we will conclude it by giving the differential equation of the third order of the implicit function y of the variable x, given by the equation $u = F(x, y) = 0$. This differential equation is

$$\frac{d^3u}{dx^3}+3\frac{d^3u}{dx^2dy}\frac{dy}{dx}+3\frac{d^3u}{dxdy^2}\left(\frac{dy}{dx}\right)^2+\frac{d^3u}{dy^3}\left(\frac{dy}{dx}\right)^3$$

$$+3\left(\frac{d^2u}{dxdy}+\frac{d^2u}{dy^2}\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+\frac{du}{dy}\frac{d^3y}{dx^3}=0\ldots\quad(4).$$

97. Suppose we have given the two simultaneous equations

$$u=F(x,y,z)=0\quad(1).$$

$$v=f(x,y,z)=0\quad(2).$$

It is theoretically possible, by combining these equations, to eliminate either variable, and get an equation expressing the relation between the other two from which the successive differential co-efficients of one of these regarded as a function of the other might be obtained. But without effecting this elimination, not always practicable, we may proceed as follows:—

Suppose x to be the independent variable, and differentiate (1) with respect to x; then (Art. 85)

$$\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}+\frac{du}{dz}\frac{dz}{dx}=0\quad(3).$$

In like manner, from (2),

$$\frac{dv}{dx}+\frac{dv}{dy}\frac{dy}{dx}+\frac{dv}{dz}\frac{dz}{dx}=0\quad(4).$$

From (3) and (4), we find

$$\frac{dy}{dx}=-\frac{\frac{du}{dx}\frac{dv}{dz}-\frac{dv}{dx}\frac{du}{dz}}{\frac{du}{dy}\frac{dv}{dz}-\frac{dv}{dy}\frac{du}{dz}}\ldots\quad(5),$$

and

$$\frac{dz}{dx}=-\frac{\frac{dv}{dx}\frac{du}{dy}-\frac{du}{dx}\frac{dv}{dy}}{\frac{dv}{dz}\frac{du}{dy}-\frac{du}{dz}\frac{dv}{dy}}\ldots\quad(6).$$

The first members of (3) and (4) are functions of x, y, z; and, by differentiating them with respect to x, we have

$$\frac{d^2u}{dx^2}+2\frac{d^2u}{dxdy}\frac{dy}{dx}+2\frac{d^2u}{dxdz}\frac{dz}{dx}+\frac{d^2u}{dy^2}\left(\frac{dy}{dx}\right)^2+2\frac{d^2u}{dydz}\frac{dy}{dx}\frac{dz}{dx}$$
$$+\frac{d^2u}{dz^2}\left(\frac{dz}{dx}\right)^2+\frac{du}{dy}\frac{d^2y}{dx^2}+\frac{du}{dz}\frac{d^2z}{dx^2}=0 \ldots (7),$$

and

$$\frac{d^2v}{dx^2}+2\frac{d^2v}{dxdy}\frac{dy}{dx}+2\frac{d^2v}{dxdz}\frac{dz}{dx}+\frac{d^2v}{dy^2}\left(\frac{dy}{dx}\right)^2+2\frac{d^2v}{dydz}\frac{dy}{dx}\frac{dz}{dx}$$
$$+\frac{d^2v}{dz^2}\left(\frac{dz}{dx}\right)^2+\frac{dv}{dy}\frac{d^2y}{dx^2}+\frac{dv}{dz}\frac{d^2z}{dx^2}=0 \ldots (8).$$

From (7) and (8), by substituting in them the values of $\frac{dy}{dx}$, $\frac{dz}{dx}$, in (5) and (6), we may deduce the values of $\frac{d^2y}{dx}$ and $\frac{d^2z}{dx^2}$. They may also be found directly by differentiating (5) and (6).

98. For an application of the methods of successive differentiation, suppose we have the single relation $u=F(x,y,z)=0$ between the three variables x, y, z; then z may be considered as an implicit function of the two independent variables x, y.

It is required to find the first and second orders of the partial differential co-efficients of z with respect to x and y without solving the equation $u=F(x,y,z)=0$.

The first partial derived equation with respect to x is (Art. 85),

$$\frac{du}{dx}+\frac{du}{dz}\frac{dz}{dx}=0 \quad (1),$$

and that with respect to y is

$$\frac{du}{dy}+\frac{du}{dz}\frac{dz}{dy}=0 \quad (2);$$

in which $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, are the partial differential co-efficients of u, taken on the supposition that the variable x, y, or z, to which they separately relate, alone varies.

Eqs. 1 and 2 will give $\frac{dz}{dx}, \frac{dz}{dy}$. Differentiating (1) with respect to x, and (2) with respect to y, and also either (1) with respect to y, or (2) with respect to x, we get

$$\frac{d^2u}{dx^2} + 2\frac{d^2u}{dxdz}\frac{dz}{dx} + \frac{d^2u}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{du}{dz}\frac{d^2z}{dx^2} = 0 \quad (3),$$

$$\frac{d^2u}{dy^2} + 2\frac{d^2u}{dydz}\frac{dz}{dy} + \frac{d^2u}{dz^2}\left(\frac{dz}{dy}\right)^2 + \frac{du}{dz}\frac{d^2z}{dy^2} = 0 \quad (4),$$

$$\frac{d^2u}{dxdy} + \frac{d^2u}{dzdx}\frac{dz}{dy} + \frac{d^2u}{dzdy}\frac{dz}{dx} + \frac{d^2u}{dz^2}\frac{dz}{dy}\frac{dz}{dx} + \frac{du}{dz}\frac{d^2z}{dxdy} = 0 \quad (5);$$

and from the five Eqs. 1, 2, 3, 4, and 5, we can deduce $\frac{dz}{dx}, \frac{dz}{dy}, \frac{d^2z}{dx^2}, \frac{d^2z}{dy^2}, \frac{d^2z}{dxdy}$.

Ex. 1. Given $y^3 + x^3 - 3axy = 0$, to find the value of $\frac{d^2y}{dx^2}$.

The first differential equation is

$$y^2\frac{dy}{dx} + x^2 - ay - ax\frac{dy}{dx} = 0 \quad (1);$$

and the second,

$$(y^2 - ax)\left(\frac{d^2y}{dx^2}\right) - 2a\frac{dy}{dx} + 2y\left(\frac{dy}{dx}\right)^2 + 2x = 0 \quad (2).$$

Substituting in (2) the value of $\frac{dy}{dx}$ taken from (1), we find, after a little reduction,

$$(y^2 - ax)^3\frac{d^2y}{dx^2} - 2a(ay - x^2)(y^2 - ax) + 2y(ay - x^2)^2 + 2x(y^2 - ax)^2 = 0:$$

whence

$$\frac{d^2y}{dx^2} = \frac{2a(ay - x^2)(y^2 - ax) - 2y(ay - x^2)^2 - 2x(y^2 - ax)^2}{(y^2 - ax)^3};$$

and this, after performing the operations indicated in the numerator of the second member, and reducing by the given equation, becomes

$$\frac{d^2y}{dx^2} = -\frac{3a^3xy}{(y^2 - ax)^3}.$$

Ex. 2. Given $b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2 - a^2b^2c^2 = 0$, to find $\frac{d^2z}{dx^2}, \frac{d^2z}{dy^2}, \frac{d^2z}{dxdy}$,

$$\frac{d^2z}{dx^2} = -\frac{c^2(a^2z^2 + c^2x^2)}{a^4z^3}, \quad \frac{d^2z}{dy^2} = -\frac{c^2(b^2z^2 + c^2y^2)}{b^4z^3},$$

$$\frac{d^2z}{dxdy} = -\frac{c^4xy}{a^2b^2z^3}.$$

SECTION X.

INVESTIGATION OF THE TRUE VALUE OF EXPRESSIONS WHICH PRESENT THEMSELVES UNDER FORMS OF INDETERMINATION.

99. It sometimes happens that the expressions under consideration assume, for particular values of the variable or variables involved, some one of the forms $\frac{0}{0}$, $\pm\frac{\infty}{\infty}$, $0\times\infty$, 0^0, ∞^0, $\pm 1^\infty$, $\infty-\infty$, called forms of indetermination, though the value of the expressions may be determinate. Our object now is to establish the rules by which may be found the true value of an expression which reduces to any one of these forms.

100. ***Of the Form*** $\frac{0}{0}$. This form can only result from a fraction in the numerator and denominator of which there is a common factor, which factor becomes zero for the particular values of the variable or variables which reduce the expression to $\frac{0}{0}$. Thus take the fraction $\frac{P}{Q}\frac{(x-a)^m}{(x-a)^n}$, in which P and Q may or may not be functions of x; but, if they are, they do not contain the factor $x-a$, and therefore do not become zero when $x=a$. If in this fraction, as it stands, we make $x=a$, it takes the form $\frac{0}{0}$; but if, before giving x this value, the fraction be written $\frac{P}{Q}(x-a)^{m-n}$, it is seen that the true value of the fraction for $x=a$ is 0 if $m>n$, ∞ if $m<n$, and $\frac{P}{Q}$ if $m=n$. This suggests the following rule for the evaluation

of expressions which take this form; viz., discover, if possible, the factors common to the numerator and denominator of the fraction, and divide them out. What the result reduces to by giving the variables their assigned values is the true value of the expression.

Example. $\dfrac{x^3+x^2-3x-3}{x^3-2x^2-3x+6}=\dfrac{0}{0}$ when $x=\sqrt{3}$;

but
$$\frac{x^3+x^2-3x-3}{x^3-2x^2-3x+6}=\frac{(x^2-3)(x+1)}{(x^2-3)(x-2)}=\frac{x+1}{x-2}$$
$$=-\frac{1+\sqrt{3}}{2-\sqrt{3}} \text{ for } x=\sqrt{3}.$$

Many cases of the form $\frac{0}{0}$ may be treated as follows:—

Take the fraction $\dfrac{(x-a)^{\frac{1}{2}}}{(x^2-a^2)^{\frac{1}{3}}}$, which becomes $\frac{0}{0}$ when $x=a$.

Make $x=a+h$; then
$$\frac{(a+h-a)^{\frac{1}{2}}}{(a^2+2ah+h^2-a^2)^{\frac{1}{3}}}=\frac{h^{\frac{1}{2}}}{h^{\frac{1}{3}}(2a+h)^{\frac{1}{3}}}=\frac{h^{\frac{1}{6}}}{(2a+h)^{\frac{1}{3}}}=0 \text{ when}$$
$h=0$, which corresponds to $x=a$.

Also the fraction $\dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}=\dfrac{0}{0}$ for $x=a$; making $x=a+h$,
$$\frac{\sqrt{a+h}-\sqrt{a}+\sqrt{h}}{\sqrt{2ah+h^2}}=\frac{\sqrt{a+h}-(\sqrt{a}-\sqrt{h})}{\sqrt{2ah+h^2}}.$$
Multiplying numerator and denominator of this by $\sqrt{a+h}+(\sqrt{a}-\sqrt{h})$, we find
$$\frac{2\sqrt{ah}}{(2ah+h^2)^{\frac{1}{2}}\left((a+h)^{\frac{1}{2}}+(\sqrt{a}-\sqrt{h})\right)}=\frac{1}{\sqrt{2a}},$$
for $h=0$, after dividing out the common factor $h^{\frac{1}{2}}$.

The examples already given have been solved by common algebraic transformations; but most of the cases which present

themselves can be more easily solved by means of the differential calculus.

101. Suppose the fraction to be $\frac{F(x)}{f(x)}$, and that both $F(x)$ and $f(x)$, as also their successive differential co-efficients up to the $(n-1)^{\text{th}}$ order inclusively, vanish for $x = a$; then it has been proved (Art. 56) that

$$\frac{F(a+h)}{f(a+h)} = \frac{F^{(n)}(a+\theta h)}{f^{(n)}(a+\theta h)};$$

and consequently, by making $h = 0$, we have

$$\frac{F(a)}{f(a)} = \frac{F^{(n)}(a)}{f^{(n)}(a)}.$$

Hence, to obtain the true value of the vanishing fraction $\frac{F(x)}{f(x)}$ when $x = a$, form the successive differential co-efficients of both terms of the given fraction until one is found, whether of numerator or denominator, that does not vanish for $x = a$; and take the value, when $x = a$, of the fraction whose terms are respectively the differential co-efficients, of the order of that thus found, of the corresponding terms of the given fraction.

If one of these differential co-efficients vanishes, the value of the fraction will be 0 or ∞, according as it is that of the numerator or of the denominator; and it will be finite if the first of the differential co-efficients that do not vanish is of the same order in the two terms of the fraction.

Ex. 1. $$\frac{e^x - e^{-x}}{\sin. x} = \frac{0}{0} \text{ for } x = 0.$$

$$F(x) = e^x - e^{-x},\ F'(x) = e^x + e^{-x},\ f(x) = \sin. x,$$
$$f'(x) = \cos. x,$$
$$\left(\frac{F(x)}{f(x)}\right)_{x=0} = \left(\frac{F'(x)}{f'(x)}\right)_{x=0} = \left(\frac{e^x + e^{-x}}{\cos. x}\right)_{x=0} = 2.$$

Ex. 2. $\left(\frac{x - \sin. x}{x^3}\right)_{x=0} = \left(\frac{1 - \cos. x}{3x^2}\right)_{x=0} = \left(\frac{\sin. x}{6x}\right)_{x=0}$

$$= \left(\frac{\cos. x}{6}\right)_{x=0} = \frac{1}{6}.$$

Ex. 3. $\left(\frac{lx}{x-1}\right)_{x=1} = \left(\frac{1}{x}\right)_{x=1} = 1.$

102. Form $\frac{\infty}{\infty}$. If the two functions $F(x)$, $f(x)$, become infinite for $x = a$, the fraction $\frac{F(x)}{f(x)}$ reduces to $\frac{\infty}{\infty}$. But in this case the fraction may be put under the form $\frac{\frac{1}{f(x)}}{\frac{1}{F(x)}}$, which, for $x = a$, becomes $\frac{0}{0}$, and may therefore be treated by the preceding rule. Thus

$$\frac{F(a)}{f(a)} = \frac{\frac{1}{f(a)}}{\frac{1}{F(a)}} = \frac{\frac{f'(a)}{\left(f(a)\right)^2}}{\frac{F'(a)}{\left(F(a)\right)^2}} = \frac{\left(F(a)\right)^2}{\left(f(a)\right)^2}\frac{f'(a)}{F'(a)}:$$

whence $\frac{F(a)}{f(a)} = \frac{F'(a)}{f'(a)}$; and the true value of the ratio $\frac{F(a)}{f(a)} = \frac{\infty}{\infty}$ is the value of $\frac{F'(a)}{f'(a)}$.

If all the differential co-efficients of both terms of the fraction become infinite up to the $(n-1)^{\text{th}}$ order inclusively, then

$$\frac{F(a)}{f(a)} = \frac{F'(a)}{f'(a)} = \frac{F''(a)}{f''(a)} = \cdots = \frac{F^{(n)}(a)}{f^{(n)}(a)};$$

and the true value of a ratio, that, for a particular value of the variable, takes the form $\frac{\infty}{\infty}$, is the value of the ratio of the differential co-efficients of the order of that first found, whether

of numerator or denominator, which does not become infinite for the assigned value of the variable.

Example. For $x = 0$,

$$\frac{l\frac{1}{x}}{\text{cosec.}\, x} = \frac{0}{0} = \frac{\sin.^2 x}{x \cos.\, x} = \frac{2 \sin.\, x \cos.\, x}{\cos.\, x - x \sin.\, x} = 0.$$

103. The rules which have been given for finding the true value of ratios which take the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are applicable for infinite as well as for finite values of the variable. This follows from the fact, that the reasoning by which these rules were established requires only that the value attributed to x, causing the fraction to assume the one or the other of the above forms, should be the same in both terms, but does not involve any supposition in regard to the magnitude of this value. The rule depending on differentiation may be demonstrated directly when the form of indetermination comes from the hypothesis $x = \infty$.

Represent the terms of the fraction by $F(x)$, $f(x)$, as before, and suppose, that, for $x = \infty$, we have either $F(x) = 0, f(x) = 0$, or $F(x) = \infty$, $f(x) = \infty$; then, putting $\frac{1}{y}$ for x,

$$\frac{F(x)}{f(x)} = \frac{F\left(\frac{1}{y}\right)}{f\left(\frac{1}{y}\right)}, \text{ and } \frac{F'(x)}{f'(x)} = \frac{\frac{1}{y^2}\, F'\left(\frac{1}{y}\right)}{\frac{1}{y^2}\, f'\left(\frac{1}{y}\right)} = \frac{F'\left(\frac{1}{y}\right)}{f'\left(\frac{1}{y}\right)};$$

but, by rules already given,

$$\left\{\frac{F\left(\frac{1}{y}\right)}{f\left(\frac{1}{y}\right)}\right\}_{y=0} = \left\{\frac{\frac{1}{y^2}\, F'\left(\frac{1}{y}\right)}{\frac{1}{y^2}\, f'\left(\frac{1}{y}\right)}\right\}_{y=0} = \left\{\frac{F'\left(\frac{1}{y}\right)}{f'\left(\frac{1}{y}\right)}\right\}_{y=0}:$$

hence

$$\left(\frac{F(x)}{f(x)}\right)_{x=\infty} = \left\{\frac{F\left(\frac{1}{y}\right)}{f\left(\frac{1}{y}\right)}\right\}_{y=0} = \left\{\frac{F'\left(\frac{1}{y}\right)}{f'\left(\frac{1}{y}\right)}\right\}_{y=0} = \left(\frac{F'(x)}{f'(x)}\right)_{x=\infty}.$$

Ex. 1. For $x=\infty$ we have, when $a > 1$,

$$\frac{a^x}{x} = \frac{a^x la}{1} = \infty .$$

Ex. 2. When $x=\infty$,

$$\frac{Lx}{x} = \frac{1}{xla} = 0.$$

Ex. 3. When $x=\infty$, and n is the integer which immediately follows a,

$$\frac{x^a}{e^x} = \frac{a(a-1)(a-2)\ldots(a-n+1)}{x^{n-a}e^x} = 0.$$

104. Form $0\times\infty$. Let $F(x), f(x)$, be two functions of x, one of which becomes 0, and the other infinity, for a particular value attributed to x. For $x=a$, suppose $F(x)=0, f(x)=\infty$. The product may be put under the forms $u = F(x)\times f(x) = \dfrac{F(x)}{\frac{1}{f(x)}} = \dfrac{f(x)}{\frac{1}{F(x)}}$; the last two of which, for the assigned value of x, take respectively the forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, and can therefore be treated by the preceding rules.

Ex. 1. $\quad u = l\left(2-\frac{x}{a}\right)\times \tan.\frac{\pi x}{2a} = 0\times\infty$ when $x=a$

But $\quad l\left(2-\frac{x}{a}\right)\times\tan.\frac{\pi x}{2a} = \dfrac{l\left(2-\frac{x}{a}\right)}{\dfrac{1}{\tan.\frac{\pi x}{2a}}} = \dfrac{l\left(2-\frac{x}{a}\right)}{\cot.\frac{\pi x}{2a}}$,

and $$\left\{\frac{l\left(2-\frac{x}{a}\right)}{\cot.\frac{\pi x}{2a}}\right\}_{x=a}=\frac{0}{0}=\left\{\frac{-\frac{1}{a}\frac{1}{2-\frac{x}{a}}}{-\frac{\pi}{2a}\frac{1}{\sin.^2\frac{\pi x}{2a}}}\right\}_{x=a}=\frac{2}{\pi}.$$

Ex. 2. For $x=0$.

$$xlx=-0\times\infty,$$

$$xlx=\frac{lx}{x^{-1}}, \text{ and } \left(\frac{lx}{x^{-1}}\right)_{x=0}=\left(-\frac{x^{-1}}{x^{-2}}\right)_{x=0}=0.$$

Ex. 3. $x^m(lx)^n=0\times\infty$ for $x=0$, when the exponents m and n are positive.

Make $x=\frac{1}{e^y}$, then $x^m(lx)^n=(-1)^n\frac{y^n}{e^{my}}$. This, by Ex. 3, Art. 103, is zero when $y=\infty$, which answers to $x=0$.

105. *Forms* 0^0, ∞^0, $\pm 1^\infty$.

In the explicit function $y=\left(F(x)\right)^{f(x)}$ of the variable x, suppose that $F(x), f(x)$, are such, that, for the particular value $x=a$, y assumes any one of the above forms; then, to deduce a rule for the evaluation of y, we proceed thus: —

Take the Napierian logarithms of both members of the equation $y=\left(F(x)\right)^{f(x)}$, and we have

$$ly=f(x)\,lF(x)=\frac{lF(x)}{\frac{1}{f(x)}}.$$

Now, since, to have one of the proposed forms, $F(x)$, for the assigned value of x, must take one of the values 0, ∞, or 1, $lF(x)$ will become either $-\infty$, $+\infty$, or 0, and $\dfrac{lF(x)}{\frac{1}{f(x)}}$ will take one or the other of the forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, and may therefore be

used for calculating the true value of ly, from which we pass to that of the function itself.

Ex. 1. x^x for $x = 0$ becomes 0^0. In this case, $\dfrac{lF(x)}{\dfrac{1}{f(x)}} = \dfrac{lx}{\dfrac{1}{x}}$, which, for $x = 0$, is equal to

$$\left\{\frac{\dfrac{1}{x}}{-\dfrac{1}{x^2}}\right\}_{x=0} = 0: \therefore ly = 0, y = 1.$$

Ex. 2. $x^{\frac{1}{x}} = \infty^0$ when $x = \infty$.

Here $\dfrac{lF(x)}{\dfrac{1}{f(x)}} = \dfrac{lx}{x}$; and this, when $x = \infty$, $= \dfrac{1}{x} = 0$:

$$\therefore ly = 0, y = 1.$$

Ex. 3.
$$x^{\frac{1}{1-x}} = 1^\infty \text{ when } x = 1,$$
$$\frac{lF(x)}{\dfrac{1}{fx}} = \frac{lx}{1-x} = \frac{0}{0} \text{ when } x = 1,$$
$$ly = -1: \therefore y = e^{-1}.$$

106. Form $\infty - \infty$.

If the functions $F(x)$, $f(x)$, of x, both become infinite when $x = a$, then, for this value,

$$F(x) - f(x) = \infty - \infty.$$

To deduce a rule for the evaluation of expressions that take this form, make $F(x) = \dfrac{1}{F_1(x)}$, $f(x) = \dfrac{1}{f_1(x)}$; then the value of x that causes $F(x), f(x)$, to become infinite, must reduce $F_1(x)$, $f_1(x)$, to zero; and, if a be this value of x, we have

$$F(a) - f(a) = \frac{1}{F_1(a)} - \frac{1}{f_1(a)} = \frac{f_1(a) - F_1(a)}{F_1(a)f_1(a)} = \frac{0}{0},$$

and the case is thus made to fall under the rule of Art. 101.

Ex. 1. $\quad$ Sec. $x -$ tan. $x = \infty - \infty$ when $x = \frac{\pi}{2}$,

$$\text{sec. } x - \text{tan. } x = \frac{1}{\text{cos. } x} - \frac{\text{sin. } x}{\text{cos. } x} = \frac{1 - \text{sin. } x}{\text{cos. } x},$$

and
$$\left(\frac{1 - \text{sin. } x}{\text{cos. } x.}\right)_{x=\frac{\pi}{2}} = \left(\frac{\text{cos. } x}{\text{sin. } x}\right)_{x=\frac{\pi}{2}} = 0.$$

Ex. 2. $\quad \frac{1}{lx} - \frac{x}{lx} = \infty - \infty$ when $x = 1$,

$$\left(\frac{1}{lx} - \frac{x}{lx}\right)_{x=1} = \left(\frac{1-x}{lx}\right)_{x=1} = -1.$$

107. It may happen that not only do $F(x)$, $f(x)$, in the ratio $\frac{F(x)}{f(x)}$, vanish for the assigned value of the variable, but also all their successive differential co-efficients, however far the differentiation be carried. For suppose $F(x) = a^{-\frac{1}{x^n}}$, which becomes 0 for $x = 0$ if a and n are positive, and $a > 1$; then

$$F'(x) = \frac{nla.a^{-\frac{1}{x^n}}}{x^{n+1}},$$

$$F''(x) = nla.\,a^{-\frac{1}{x^n}}\left(\frac{nla}{x^{2(n+1)}} - \frac{n+1}{x^{n+2}}\right).$$

Making $x = \frac{1}{z}$, these differential co-efficients become

$$F'(x) = \frac{nlaz^{n+1}}{a^{z^n}},$$

$$F''(x) = \frac{nla\left(nlaz^{2(n+1)} - (n+1)\,z^{n+2}\right)}{a^{z^n}}.$$

It is needless to carry the differentiation further to see that each differential co-efficient will contain a factor of the form $\frac{z^m}{a^{z^n}}$, in which a, m, and n are positive, and $a > 1$. This factor takes the form $\frac{\infty}{\infty}$ for $z = \infty$; and if we apply to it the method

for finding the true value of such expressions by differentiation, differentiating p times, p being the whole number next above m, z will disappear from the numerator of the ratio of the differential co-efficients of the order p, and this ratio would be of the form $\frac{k}{\varphi(z)}$, in which k is a constant, and $\varphi(z)$ a function of z, that becomes infinite when $z = \infty$. Therefore all the differential co-efficients of $F(x)$ vanish when $x = 0$, which answers to $z = \infty$. Hence, if $f(x) = b^{-\frac{1}{x^q}}$, the terms of the ratio $\frac{F(x)}{f(x)}$, and all their differential co-efficients, vanish for $x = 0$, if a, b, n, q, are positive, and a and b are each greater than 1. The true value of this ratio cannot then be found by the method of differentiation.

When $n = q$, the ratio becomes $\left(\frac{a}{b}\right)^{-\frac{1}{x^n}}$, the true value of which, for $x = 0$, is 0, if $a > b$, and ∞ if $a < b$.

108. The solution of cases of indetermination is often facilitated by transforming the example so as to make it take a form of indetermination different from that under which it presents itself. Thus

$$\frac{e^{-\frac{1}{x}}}{x} = \frac{\frac{1}{e^{\frac{1}{x}}}}{x} \text{ becomes } \frac{0}{0} \text{ when } x = 0;$$

but

$$\frac{e^{-\frac{1}{x}}}{x} = \frac{1}{xe^{\frac{1}{x}}} = \frac{1}{0 \times \infty} \text{ when } x = 0;$$

and the true value of the given expression is 1 divided by the true value of $xe^{\frac{1}{x}}$ when $x = 0$.

$$\left(xe^{\frac{1}{x}}\right)_{x=0} = \left\{\frac{e^{\frac{1}{x}}}{\frac{1}{x}}\right\}_{x=0} = \left\{\frac{e^{\frac{1}{x}} \times -\frac{1}{x^2}}{-\frac{1}{x^2}}\right\}_{x=0} = \left(e^{\frac{1}{x}}\right)_{x=0} = \infty :$$

$$\therefore \left(\frac{e^{-\frac{1}{x}}}{x}\right)_{x=0} = \frac{1}{\infty} = 0.$$

Again: if $F(x)$ becomes infinite when $x = \infty$, then (Art. 102)

$$\left(\frac{F(x)}{x}\right)_{x=\infty} = \Big(F'(x)\Big)_{x=\infty}$$

But (Art. 56)

$$\frac{F(x+h) - F(x)}{h} = F'(x + \theta h);$$

and it is evident, that as x increases, and finally becomes infinite, the second member of this equation converges towards and finally becomes $F'(x)$: hence

$$\left(\frac{F(x)}{x}\right)_{x=\infty} = \left(\frac{F(x+h) - F(x)}{h}\right)_{x=\infty};$$

or by making $h = 1$, as we may, since h is arbitrary,

$$\left(\frac{F(x)}{x}\right)_{x=\infty} = \Big(F(x+1) - F(x)\Big)_{x=\infty}.$$

If, now, the value of $\Big(F(x)\Big)^{\frac{1}{x}}$, when $x = \infty$, is required, we have

$$\Big(F(x)\Big)^{\frac{1}{x}} = e^{\frac{lF(x)}{x}} \quad (1);$$

a true equation, as may be seen by taking the logarithms of both members: therefore

$$\Big(F(x)\Big)^{\frac{1}{x}}_{x=\infty} = \left(e^{\frac{lF(x)}{x}}\right)_{x=\infty} \quad (2);$$

and the proposition is thus reduced to the evaluation of

$e^{\frac{lF(x)}{x}}$, or rather of $\frac{lF(x)}{x}$ when $x = \infty$.

By what is proved above,

$$\left(\frac{lF(x)}{x}\right)_{x=\infty} = \left(lF(x+1) - lF(x)\right)_{x=\infty} = \left(l\frac{F(x+1)}{F(x)}\right)_{x=\infty} \quad (3).$$

But, from Eq. 1, we have

$$l\left(F(x)\right)^{\frac{1}{x}} = \frac{lF(x)}{x}:$$

therefore, by Eq. 3,

$$\left(F(x)\right)^{\frac{1}{x}}_{x=\infty} = \left(\frac{F(x+1)}{F(x)}\right)_{x=\infty}.$$

Let this be applied to the determination of the true value of

$$\left(\frac{x^x}{1.2\ldots x}\right)^{\frac{1}{x}} \text{ when } x=\infty.$$

Now, by what has just been proved, the required value is that of

$$\frac{(x+1)^{x+1}}{1.2\ldots(x+1)}\,\frac{1.2\ldots x}{x^x} = \left(\frac{x+1}{x}\right)^x = \left(1+\frac{1}{x}\right)^x \text{ for } x=\infty.$$

But (Art. 9) $\qquad \left(1+\dfrac{1}{x}\right)^x_{x=\infty} = e.$

109. Thus far, the discussion of the indeterminate forms has been confined to functions of a single variable. A few cases will now be considered in which these forms present themselves in functions of more than one variable. We remark, that a function of two variables may assume the form $\frac{0}{0}$, either when a particular value is attributed to but one of the variables, or when both variables have particular values given them. An example of the first case is

$$z = \frac{b(x-a)}{y(x^2-a^2)+(x-a)^2},$$

which, for $x=a$, reduces to $\frac{0}{0}$, whatever be the value of y;

but by dividing out the common factor $x-a$, and then making $x=a$, we have $z=\dfrac{b}{2ay}$.

An example of the second is

$$z=\frac{c(x-a)}{m(y-b)},$$

which takes the form $\frac{0}{0}$ for $x=a$, $y=b$, and, for these values of the variables, is really indeterminate. For let p denote the ratio $\dfrac{x-a}{y-b}$, then $z=\dfrac{cp}{m}$; and, since x and y are independent, p is an arbitrary quantity, and z is therefore indeterminate.

110. To investigate a rule for the evaluation of the indeterminate forms of functions of two or more variables, take the function $u=F(x,y)$, x and y being independent, and suppose the function to be finite and continuous for all values of x and y between $x=a$, $x=a+h$, $y=b$, $y=b+k$; and further, that all the partial differential co-efficients of the function, up to $(n-1)^{\text{th}}$ inclusively, vanish for $x=a$, $y=b$; but that those of the n^{th} order neither vanish nor become infinite for these values of x and y.

For the time, denote by ht, kt, the increments of a and b; so that the function, when the values of x and y with their respective increments are substituted, is $F(a+ht,\, b+kt)$, which becomes $F(a+h,\, b+k)$ by making $t=1$: then denoting $F(a+ht,\, b+kt)$, which is a function of t, by $f(t)$, we have

$$F(a+ht,\, b+kt)=f(t) \quad (1);$$

and, making in this $t=0$,

$$F(a,\, b)=f(0) \quad (2).$$

If $f(t)$ is finite and continuous for all values of t, from $t=0$

up to $t =$ any assigned value, $t = t$; and if, in addition, all the differential co-efficients of $f(t)$, up to the $(n-1)^{\text{th}}$ inclusively, vanish for $t = 0$, while that of the n^{th} order is neither zero nor infinite for $t = 0$; then (Art. 56)

$$f(t) - f(0) = \frac{t^n}{1.2\ldots n} f^{(n)}(\theta t) \quad (3).$$

To simplify the application of this equation to our purposes, make $x' = a + ht$, $y' = b + kt$; whence $\frac{dx'}{dt} = h$, $\frac{dy'}{dt} = k$, and $f(t) = F(x', y')$:

$$\therefore \quad f'(t) = \frac{dF}{dx'}\frac{dx'}{dt} + \frac{dF}{dy'}\frac{dy'}{dt} = \frac{dF}{dx'}h + \frac{dF}{dy'}k.$$

Making $t = 0$, observing that then $x' = a$, $y' = b$, and that what $\frac{dF}{dx'}$, $\frac{dF}{dy'}$, become, will be identically the same as what $\frac{dF}{dx}$, $\frac{dF}{dy}$, become when $x = a$, $y = b$, and denoting these differential co-efficients for this value of t by $\left(\frac{dF}{dx}\right)_0$, $\left(\frac{dF}{dy}\right)_0$, we have

$$f'(0) = \left(\frac{dF}{dx}\right)_0 h + \left(\frac{dF}{dy}\right)_0 k.$$

If $\left(\frac{dF}{dx}\right)_0$, $\left(\frac{dF}{dy}\right)_0$, both vanish, then $f'(0) = 0$, and we must proceed to the 2d differential co-efficient of $f(t)$, which is

$$f''(t) = \frac{d.f'(t)}{dt} = \frac{d^2F}{dx'^2}h^2 + 2\frac{d^2F}{dx'dy'}hk + \frac{d^2F}{dy'^2}k^2;$$

and, in this making $t = 0$, we have, by adopting a notation in harmony with that in the expression for $f'(0)$,

$$f''(0) = \left(\frac{d^2F}{dx^2}\right)_0 h^2 + 2\left(\frac{d^2F}{dxdy}\right)_0 hk + \left(\frac{d^2F}{dy^2}\right)_0 k^2;$$

and in this also $\left(\frac{d^2F}{dx^2}\right)_0 \ldots$ are what $\frac{d^2F}{dx^2}\ldots$ become when $x=a, y=b$. If all the partial differential co-efficients of the second order vanish for $t=0$, then $f''(0)=0$; and we must pass to the 3d differential co-efficient of $f(t)$, and in this make $t=0$. We should thus find

$$f'''(0)=\left(\frac{d^3F}{dx^3}\right)_0 h^3+3\left(\frac{d^3F}{dx^2dy}\right)_0 h^2k+3\left(\frac{d^3F}{dxdy^2}\right)hk^2+\frac{d^3F}{dy}k^3,$$

and so on; the expression for $f^{(n)}(t)$, all up to that vanishing for $t=0$, being

$$f^{(n)}(t)=$$

$$\left(\frac{d^nF}{dx'^n}\right)h^n+n\left(\frac{d^nF}{dx'^{n-1}dy'}\right)h^{n-1}k+\cdots+n\left(\frac{d^nF}{dx'dy'^{n-1}}\right)hk^{n-1}$$
$$+\left(\frac{d^nF}{dy'^n}\right)k^n,$$

the laws governing the co-efficients and exponents being obviously the same as in the Binomial Formula.

Now, since $F(x,y)$, $F(x',y')$, differ only by having x and y in the one replaced respectively by x' and y' in the other, it follows that any partial differential co-efficient of $F(x,y)$ will be the same function of x and y that the corresponding partial differential co-efficient of $F(x',y')$ is of x' and y'; and hence the hypothesis that renders $x=x'$, $y=y'$, will, at the same time, cause these differential co-efficients to be equal. Therefore make $x=x'=a+ht$, $y=y'=b+kt$, and we may write

$$f^{(n)}(t)=\left\{\begin{aligned}&\frac{d^nF}{dx^n}h^n+n\frac{d^nF}{dx^{n-1}dy}h^{n-1}k+\ldots\\&+n\frac{d^nF}{dxdy^{n-1}}hk^{n-1}+\frac{d^nF}{dy^n}k^n\end{aligned}\right\}_{\substack{x=a+ht\\y=b+kt}} \quad (4);$$

all of the several orders of partial differential co-efficients of $F(x, y)$, up to and exclusive of the n^{th}, vanishing for $x = a$, $y = b$, that is, for $t = 0$ in $f'(t) \ldots f^{(n-1)}(t)$; but all of those of the n^{th} order not vanishing. Then, writing θt for t in Eq. 4, and substituting in Eq. 3, we have

$$F(a + ht, b + kt) - F(a, b)$$
$$= \frac{t^n}{1.2 \ldots n}\left(\frac{d^n F}{dx^n} h^n + n \frac{d^n F}{dx^{n-1}dy} h^{n-1}k + \cdots \right.$$
$$\left. \cdots + n \frac{d^n F}{dx dy^{n-1}} hk^{n-1} + \frac{d^n F}{dy^n} k^n\right)_{\substack{x = a + \theta ht \\ y = b + \theta kt}};$$

and if, in this equation, we make $t = 1$; then

$$F(a + h, b + k) - F(a, b)$$
$$= \frac{1}{1.2.3 \ldots n}\left(\frac{d^n F}{dx^n} h^n + n \frac{d^n F}{dx^{n-1}dy} h^{n-1}k + \cdots \right.$$
$$\left. \cdots + n \frac{d^n F}{dx dy^{n-1}} hk^{n-1} + \frac{d^n F}{dy^n} k^n\right)_{\substack{x = a + \theta h \\ y = b + \theta k}} \quad (5),$$

which enunciates a theorem relating to a function of two independent variables analogous to that demonstrated in Art. 56 for a function of a single variable.

In Eq. 5, suppose both a and b to be zero, and then change h and k into y and x, as we may do, since h and k are not only independent of each other, but may have any values, and we have

$$F(x, y) - F(0, 0) = \frac{1}{1.2.3 \ldots n}\left(\frac{d^n F}{dx^n} h^n + n \frac{d^n F}{dx^{n-1}dy} h^{n-1}k + \cdots \right.$$
$$\left. \cdots + n \frac{d^n F}{dx dy^n} hk^{n-1} + \frac{d^n F}{dy^n} k^n\right)_{\substack{x = \theta h \\ y = \theta k}} \quad (6),$$

which expresses another theorem relating to a function of two independent variables similar to that in Art. 56 for a function of a single variable.

111. Let $F(x, y)$, $f(x, y)$, be two functions of the independent variables x and y, and suppose that not only the functions, but also all of their successive partial differential co-efficients, up to those of the $(n-1)^{\text{th}}$ order inclusively, vanish for $x = a$, $y = b$; but that, in respect to those of the n^{th} order, all do not vanish, nor do any of them become infinite for these values for x and y: then by Eq. 5, Art. 110, remembering that by hypothesis $F(a, b) = 0$, $f(a, b) = 0$, we have

$$F(a+h, b+k)$$
$$= \frac{1}{1.2.3\ldots n}\left(\frac{d^nF}{dx^n}h^n + n\frac{d^nF}{dx^{n-1}dy}h^{n-1}k + \cdots \right.$$
$$\left.\cdots + n\frac{d^nF}{dxdy^{n-1}}hk^{n-1} + \frac{d^nF}{dy^n}k^n\right)_{\substack{x=a+\theta h\\ y=b+\theta k}} \quad (1),$$

$$f(a+h, b+k)$$
$$= \frac{1}{1.2.3\ldots n}\left(\frac{d^nf}{dx^n}h^n + n\frac{d^nf}{dx^{n-1}dy}h^{n-1}k + \cdots \right.$$
$$\left.\cdots + n\frac{d^nf}{dxdy^{n-1}}hk^{n-1} + \frac{d^nf}{dy^n}k^n\right)_{\substack{x=a+\theta h\\ y=b+\theta k}} \quad (2).$$

Dividing (1) by (2), member by member, we have

$$\frac{F(a+h, b+k)}{f(a+h, b+k)}$$
$$= \left\{\frac{\frac{d^nF}{dx^n}h^n + n\frac{d^nF}{dx^{n-1}dy}h^{n-1}k + \cdots + n\frac{d^nF}{dxdy^{n-1}}hk^{n-1} + \frac{d^nF}{dy^n}k^n}{\frac{d^nf}{dx^n}h^n + n\frac{d^nf}{dx^{n-1}dy}h^{n-1}k + \cdots + n\frac{d^nf}{dxdy^{n-1}}hk^{n-1} + \frac{d^nf}{dy^n}k^n}\right\}_{\substack{x=a+\theta h\\ y=b+\theta k}} \quad (3).$$

Now, the increments h and k are quite arbitrary, and, like the variables to which they refer, are also independent of each other: we may therefore assume $k = mh$, in which m is an arbitrary constant.

Substituting this value for k, in Eq. 3, dividing out the factor h^n, common to the numerator and denominator of the second member, and then making $h = 0$, we have

$$\frac{F(a,b)}{f(a,b)}=\frac{0}{0}$$

$$=\left\{\frac{\frac{d^nF}{dx^n}+n\frac{d^nF}{dx^{n-1}dy}m+\cdots+n\frac{d^nF}{dxdy^{n-1}}m^{n-1}+\frac{d^nF}{dy^n}m^n}{\frac{d^nf}{dx^n}+n\frac{d^nf}{dx^{n-1}dy}m+\cdots+n\frac{d^nf}{dxdy^{n-1}}m^{n-1}+\frac{d^nf}{dy^n}m^n}\right\}_{\substack{x=a\\y=b}} \quad (4).$$

This value of $\frac{F(a,b)}{f(a,b)}=\frac{0}{0}$ is indeterminate, since m is arbitrary; and generally, if two functions of two independent variables both reduce to zero for particular values of the variables, the ratio of the functions for such values is really indeterminate.

112. Making $n=1$, in Eq. 4 of the last article, we have

$$\frac{F(a,b)}{f(a,b)}=\frac{\frac{dF}{dx}+\frac{dF}{dy}m}{\frac{df}{dx}+\frac{df}{dy}m};$$

and this value of $\frac{F(a,b)}{f(a,b)}$ becomes determinate if $\frac{dF}{dx}, \frac{df}{dx}$, both vanish for $x=a$, $y=b$; or, they remaining finite, if $\frac{dF}{dy}, \frac{df}{dy}$, both vanish for these values of x and y. The value of $\frac{F(a,b)}{f(a,b)}=\frac{0}{0}$ becomes, in the first case,

$$\frac{F(a,b)}{f(a,b)}=\frac{\frac{dF}{dy}}{\frac{df}{dy}}; \text{ and, in the second, } \frac{F(a,b)}{f(a,b)}=\frac{\frac{dF}{dx}}{\frac{df}{dx}}.$$

$\frac{F(a,b)}{f(a,b)}$ is also determinate if $\frac{\frac{dF}{dx}}{\frac{df}{dx}}=\frac{\frac{dF}{dy}}{\frac{df}{dy}}$: we should then have

$$\frac{F(a,b)}{f(a,b)}=\frac{\frac{dF}{dx}\left(\frac{df}{dx}+\frac{df}{dy}m\right)}{\frac{df}{dx}\left(\frac{df}{dx}+\frac{df}{dy}m\right)}=\frac{\frac{dF}{dx}}{\frac{df}{dx}}=\frac{\frac{dF}{dy}}{\frac{df}{dy}}.$$

If $\frac{dF}{dx}=0$, $\frac{df}{dx}=0$, $\frac{dF}{dy}=0$, $\frac{df}{dy}=0$, we make $n=2$ in Eq. 4, Art 111, and thus have

$$\frac{F(a,b)}{f(a,b)}=\frac{\frac{d^2F}{dx^2}+2\frac{d^2F}{dxdy}m+\frac{d^2F}{dy^2}m^2}{\frac{d^2f}{dx^2}+2\frac{d^2f}{dxdy}m+\frac{d^2f}{dy^2}m^2};$$

which is indeterminate, except in particular cases depending on the absolute and relative values assumed by the partial differential co-efficients for the values $x=a$, $y=b$.

Example. $z=\frac{lx+ly}{x+2y-3}=\frac{0}{0}$ when $x=1$, $y=1$.

Here $F(x,y)=lx+ly$, $f(x,y)=x+2y-3$,

$$\frac{dF}{dx}=\frac{1}{x}=1, \text{ for } x=1; \quad \frac{df}{dx}=1,$$

$$\frac{dF}{dy}=\frac{1}{y}=1, \text{ for } y=1; \quad \frac{df}{dy}=2; \text{ hence}$$

$$z=\frac{1+m}{1+2m}:$$

and therefore, for the assigned values of x and y, the function is really indeterminate, and may take any value between $+\infty$ and $-\infty$.

113. In the case of the implicit function $u=F(x,y)=0$, we have found

$$\frac{dy}{dx}=-\frac{\frac{dF}{dx}}{\frac{dF}{dy}} \quad (1).$$

Now, if $x=a$, $y=b$, are values of x and y, which, while they satisfy the given equation, at the same time make $\frac{dF}{dx}=0$,

$\frac{dF}{dy}=0$, then $\frac{dy}{dx}$ takes the indeterminate form $\frac{0}{0}$, and its true value, if determinate, must be found by the preceding method.

Differentiating numerator and denominator, we have for $x=a$, $y=b$,

$$\frac{dy}{dx}=-\frac{\frac{dF}{dx}}{\frac{dF}{dy}}=-\frac{\frac{d^2F}{dx^2}+\frac{d^2F}{dxdy}\frac{dy}{dx}}{\frac{d^2F}{dxdy}+\frac{d^2F}{dy^2}\frac{dy}{dx}} \quad (2),$$

from which we get

$$\frac{d^2F}{dy^2}\left(\frac{dy}{dx}\right)^2+2\frac{d^2F}{dxdy}\frac{dy}{dx}+\frac{d^2F}{dx^2}=0 \quad (3),$$

a quadratic with respect to $\frac{dy}{dx}$. This equation agrees with Eq. 3, Art. 96, observing that by supposition $\frac{dF}{dy}=0$. It must be remembered that Eqs. 2 and 3 are true only for the particular values $x=a$, $y=b$. When these values of x and y, in addition to making the function and its first partial differential co-efficients equal to 0, also make

$$\frac{d^2F}{dx^2}=0,\ \frac{d^2F}{dxdy}=0,\ \frac{d^2F}{dy^2}=0,$$

the value of $\frac{dy}{dx}$, as given by Eq. 2, again takes the form $\frac{0}{0}$; and we must in that case effect a third differentiation, which gives

$$\frac{dy}{dx}=-\frac{\frac{d^3F}{dx^3}+2\frac{d^3F}{dx^2dy}\frac{dy}{dx}+\frac{d^3F}{dxdy^2}\left(\frac{dy}{dx}\right)^2+\frac{d^2F}{dxdy}\frac{d^2y}{dx^2}}{\frac{d^3F}{dx^2dy}+2\frac{d^3F}{dxdy^2}\frac{dy}{dx}+\frac{d^3F}{dy^3}\left(\frac{dy}{dx}\right)^2+\frac{d^2F}{dy^2}\frac{d^2y}{dx^2}} \quad (4);$$

and from this, observing that by hypothesis $\frac{d^2F}{dxdy}=0$, $\frac{d^2F}{dy^2}=0$, we derive the cubic equation,

$$\frac{d^3F}{dy^3}\left(\frac{dy}{dx}\right)^3 + 3\,\frac{d^3F}{dxdy^2}\left(\frac{dy}{dx}\right)^2 + 3\,\frac{d^3F}{dx^2dy}\,\frac{dy}{dx} + \frac{d^3F}{dx^3} = 0 \quad (5).$$

Ex. 1. Determine the value of $\frac{dy}{dx}$ from $ax^2 - y^3 - by^2 = 0$ when $x = 0$, $y = 0$.

Here $$\frac{dy}{dx} = \frac{2ax}{3y^2 + 2by} = \frac{0}{0} \text{ for } x = 0,\ y = 0.$$

But, for these values of x and y, we have

$$\frac{dy}{dx} = \frac{2ax}{3y^2 + 2by} = \frac{2a}{6y\,\frac{dy}{dx} + 2b\,\frac{dy}{dx}} = \frac{2a}{2b\,\frac{dy}{dx}} \text{ for } x = 0,\ y = 0:$$

$$\therefore\ \frac{dy}{dx} = \frac{a}{b\,\frac{dy}{dx}},\quad \left(\frac{dy}{dx}\right)^2 = \frac{a}{b},\quad \frac{dy}{dx} = \pm\sqrt{\frac{a}{b}}.$$

Ex. 2. If $u = x^4 + 3a^2x^2 - 4a^2xy - a^2y^2 = 0$, find the value of $\frac{dy}{dx}$ for $x = 0$, $y = 0$. We have

$$\frac{du}{dx} = 4x^3 + 6a^2x - 4a^2y.$$

$$\frac{du}{dy} = -\,4a^2x - 2a^2y:$$

$$\therefore\quad \frac{dy}{dx} = \frac{4x^3 + 6a^2x - 4a^2y}{4a^2x + 2a^2y} = \frac{2x^3 + 3a^2x - 2a^2y}{2a^2x + a^2y}$$

$$= \frac{0}{0} \text{ for } x = 0,\ y = 0.$$

Differentiating both numerator and denominator with respect to x and y, we get

$$\frac{dy}{dx} = \frac{6x^2 + 3a^2 - 2a^2\,\frac{dy}{dx}}{2a^2 + a^2\,\frac{dy}{dx}} = \frac{3a^2 - 2a^2\,\frac{dy}{dx}}{2a^2 + a^2\,\frac{dy}{dx}} \text{ for } x = 0,\ y = 0,$$

$$= \frac{3 - 2\,\frac{dy}{dx}}{2 + \frac{dy}{dx}};$$

$$\therefore \qquad 2\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2=3-2\frac{dy}{dx},$$

or

$$\left(\frac{dy}{dx}\right)^2+4\frac{dy}{dx}-3=0;$$

$$\therefore \qquad \frac{dy}{dx}=-2\pm\sqrt{7}.$$

Should it happen that the particular values of x and y reduce $\frac{d^2F}{dy^2}$ to zero, while $\frac{d^2F}{dxdy}$, $\frac{d^2F}{dx^2}$, remain finite for these values, then Eq. 3 of this article gives, for one of the values of $\frac{dy}{dx}$,

$$\frac{dy}{dx}=-\frac{\frac{d^2F}{dx^2}}{2\frac{d^2F}{dxdy}},$$

which is finite, while the other value of $\frac{dy}{dx}$ becomes infinite, as may be shown by discussing the equation $ax^2+bx+c=0$, under the suppositions that $a=0$, and that b and c are finite.

114. The investigation of the true value of $\frac{dy}{dx}$, when it takes the form $\frac{0}{0}$ for $x=0$, $y=0$, may be simplified by the consideration, that, in this case, $\left(\frac{dy}{dx}\right)_{\substack{x=0\\y=0}}=\left(\frac{y}{x}\right)_{\substack{x=0\\y=0}}$, as is evident from the definition of differential co-efficients. Take Ex. 2 of the preceding article, and divide through by x^2; then

$$x^2+3a^2-4a^2\frac{y}{x}-a^2\left(\frac{y}{x}\right)^2=0.$$

Solving this equation with reference to $\frac{y}{x}$, and then making $x=0$, we find, as before,

$$\frac{dy}{dx}=\frac{y}{x}=-2\pm\sqrt{7}.$$

In like manner, the example

$$x^4 + ay^3 - 2axy^2 - 3ax^2y = 0,$$

which gives

$$\frac{dy}{dx} = \frac{4x^3 - 6axy - 2ay^2}{3ax^2 + 4axy - 3ay^2} = \frac{0}{0} \text{ for } x = 0,\ y = 0,$$

by dividing through by x^3, takes the form

$$x + a\left(\frac{y}{x}\right)^3 - 2a\left(\frac{y}{x}\right)^2 - 3a\frac{y}{x} = 0;$$

a cubic equation, from which, after making $x = 0$, we get for $\frac{y}{x}$ the three values 0, 3, and -1.

For another example, take the equation

$$x^4 + ax^2y + bxy^2 - y^4 = 0;$$

whence $$x + a\frac{y}{x} + b\left(\frac{y}{x}\right)^2 - y\left(\frac{y}{x}\right)^3 = 0,$$

which, for $x = 0$, $y = 0$, reduces to

$$a\frac{y}{x} + b\left(\frac{y}{x}\right)^2 = 0:$$

and therefore we have $\frac{y}{x} = 0$, and $\frac{y}{x} = -\frac{a}{b}$.

By dividing through by y^3, the assumed equation becomes

$$x\left(\frac{x}{y}\right)^3 + a\left(\frac{x}{y}\right)^2 + b\frac{x}{y} - y = 0;$$

which is satisfied by making simultaneously $x = 0$, $y = 0$, $\frac{x}{y} = 0$: hence $\frac{x}{y} = 0$, or $\frac{y}{x} = \infty$, will satisfy the given equation in connection with the values $x = 0$, $y = 0$. Therefore, when x and y have these values, $\frac{y}{x}$ may have the three values, $0, -\frac{a}{b}, \infty$.

EXAMPLES.

1. $$\left(\frac{x-1}{x^n-1}\right)_{x=1}$$ Ans. $\frac{1}{n}$.

2. $$\left(\frac{x-\sin.^{-1}x}{\sin.^3 x}\right)_{x=0}$$ Ans. $-\frac{1}{6}$.

3. $$\left(\frac{e^x-2\sin.x-e^{-x}}{x-\sin.x}\right)_{x=0}$$ Ans. 4.

4. $$\left(\frac{e^{-\frac{1}{x^2}}}{x^{2n}}\right)_{x=0}$$ Ans. 0.

In solving this example, begin by making $x^2=\frac{1}{z}$; whence $z=\infty$ when $x=0$; and we conclude that $e^{-\frac{1}{x^2}}$ decreases more rapidly as x decreases than does x^{2n}, however great be the value of n.

5. $$\left(\frac{1-x+lx}{1-\sqrt{2x-x^2}}\right)_{x=1}$$ Ans. -1.

6. $$\left(\frac{a_1^{nx}+a_2^{nx}+a_3^{nx}+\cdots+a_n^{nx}}{n}\right)^{\frac{1}{x}}_{x=0}$$ Ans. $a_1a_2a_3\ldots a_n$.

7. $$\left(\frac{x^{\frac{3}{2}}-1+(x-1)^{\frac{3}{2}}}{\sqrt{x^2-1}}\right)_{x=1}$$ Ans. 0.

8. $$\left(2^x\sin.\frac{a}{2^x}\right)_{x=\infty}$$ Ans. a.

9. $$\left(\frac{x^2}{1-\cos.nx}\right)_{x=0}$$ Ans. $\frac{2}{n^2}$.

10. $$\left(1+\frac{1}{x}\right)^x_{x=0}$$ Ans. 1.

SECTION XI.

DETERMINATION OF THE MAXIMA AND MINIMA VALUES OF FUNCTIONS OF ONE VARIABLE.

115. WHEN the value of a function, for particular values of the variables, is greater than those given by values of the variables immediately preceding or following such particular values, the function is said to be a *maximum:* when it is less, it is a *minimum.* To fix attention, suppose $y = f(x)$ to be such a function of x, that, as x gradually changes from a specific value to another, y undergoes continuous changes; but, having increased up to a certain value, then begins to decrease, or, having decreased to a certain value, then begins to increase. The value of y at the point where, from increasing, it begins to decrease, is a *maximum;* and at the point where, from decreasing, it begins to increase, it is a *minimum.* In the first case, the value of y is greater, and in the second case less, than those which immediately precede and follow. The terms *maximum* and *minimum* must be understood as relative rather than absolute; for it is plain that a function may have several maxima and minima as above defined.

Confining ourselves, for the present, to explicit functions of a single variable, we have seen (Art. 52) that such function can pass from increasing to decreasing, or the reverse, only when the first differential co-efficient of the function passes through 0 or ∞, or when this differential co-efficient changes from positive to negative, or from negative to positive.

Hence those values of x which render $y=f(x)$ a maximum or a minimum must be found among those which satisfy the equations $f'(x)=0$, $f'(x)=\infty$.

Let $x=a$ be a root of one of these equations, and let h be a very small quantity; then $f(a)$ will be a maximum if $f'(a-h)$ is positive, and $f'(a+h)$ is negative; but $f(a)$ will be a minimum if $f'(a-h)$ is negative, and $f'(a+h)$ is positive. When $f'(a-h)$, $f'(a+h)$, are both of the same sign, whether positive or negative, $f(a)$ is neither a maximum nor a minimum.

Ex. 1. $y=f(x)=2ax-x^2$,

$$f'(x)=a-x;\ f'(x)=0 \text{ gives } x=a,$$
$$f'(a-h)=a-a+h=+h,$$
$$f'(a+h)=a-a-h=-h.$$

Hence $x=a$ renders the expression $2ax-x^2$ a maximum, as may be easily verified; for, making $x=a$, $2ax-x^2$ reduces to a^2; but making $x=a+h$, or $x=a-h$, our result in either case is $a^2-h^2<a^2$.

116. The method just given for deciding whether or not a root of the equations $f'(x)=0$, $f'(x)=\infty$, answers to a maximum or minimum state of $f(x)$, is general; but, in respect to the roots of the equation $f'(x)=0$, we may for this purpose deduce a rule, that, in many cases, admits of easier application.

As before, let $x=a$ be a root of $f'(x)=0$, and suppose that $f^{(n)}(x)$ is the first among the derivatives of $f(x)$ that does not vanish for this value of x; then (Art. 56)

$$f(a+h)-f(a)=\frac{h^n}{1.2...n}f^{(n)}(a+\theta h) \quad (1).$$

Since θ is a proper fraction, and h, as we shall suppose it to be, is a very small quantity, it is obvious that the sign of $f^{(n)}(a+\theta h)$ cannot change with that of h, and is therefore

invariable: hence the sign of the second member of Eq. 1 depends on that of h^n when combined with that of $f^{(n)}(a+\theta h)$. But, if n is an even number, the sign of h^n is positive, whatever be the sign of h; and in this case the sign of the second member of Eq. 1, and consequently that of $f(a+h)-f(a)$, will be the same as that of $f^{(n)}(a+\theta h)$, or as that of $f^{(n)}(a)$, since $f^{(n)}(a+\theta h)$ and $f^{(n)}(a)$ have the same sign. If, then, n being even, $f^{(n)}(a)$ is positive, $f(a+h)-f(a)$ is also positive, whether h be positive or negative, which requires that $f(a)$ be less than $f(a \pm h)$; that is, $f(x)_{x=a}=f(a)$ is less than those values of $f(x)$ which are in the immediate vicinity of this particular value. This condition indicates a minimum state of the function. But, n being still an even number, if $f^{(n)}(a)$ is negative, then $f(a \pm h)-f(a)$ is negative, which requires that $f(a \pm h)$ be less than $f(a)$; and a maximum state of the function is indicated.

The hypothesis in respect to h being continued, if n be an odd number, then, since $(+h)^n$ and $(-h)^n$ have opposite signs, and the sign of $f^n(a+\theta h)$ does not change with that of h, the second member of Eq. 1 will change its sign as h changes from positive to negative, or the reverse. Hence $f(a+h)-f(a)$ and $f(a-h)-f(a)$ must have opposite signs, and $f(a)$ is greater than one of the expressions $f(a+h)$, $f(a-h)$, and less than the other; that is, $f(a)$ is neither greater than both the immediately preceding and immediately following values of the function, nor less than both these values; and therefore, in this case, $x=a$ renders the function neither a maximum nor a minimum. Whence the rule for deciding which of the roots of $f'(x)=0$ corresponds to maxima or to minima of $f(x)$.

"Substitute the root under consideration, in the successive derivatives of the function, until one is found that does not

vanish. If this derivative is of an even order, and the result of the substitution is positive, the root will render the function a minimum; but, if the result of the substitution is negative, the root will correspond to a maximum. If the first of the derivatives of the given function that does not vanish is of an odd order, the root corresponds to neither a maximum nor to a minimum state of the function."

Ex. 1. Find the values of x that will render

$$f(x) = x^3 - 9x^2 + 24x - 7$$

a maximum or a minimum.

$$f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8),$$
$$f''(x) = 6(x - 3).$$

From $f'(x) = 3(x^2 - 6x + 8) = 0$, we find $x = 2$, or $x = 4$. When $x = 2$, $f''(x) = 6(x - 3) = -6$; and hence, for $x = 2$, the function is a maximum. When $x = 4$, $f''(x) = +6$; and $x = 4$ renders the function a minimum.

117. When y is an implicit function of x given by the equation $F(x, y) = 0$, and the values of x corresponding to the maxima or minima values of y are required, we may, in cases in which the resolution of the equation with respect to y is possible, employ the methods of the preceding articles. But, without solving the given equation, we may proceed as follows: —

$$\text{Let } u = F(x, y) = 0;$$

then (Art. 84)

$$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}}.$$

Limiting our discussion to the values of x derived from the equation $\frac{dy}{dx} = 0$, if $\frac{du}{dy}$ is finite, $\frac{dy}{dx} = 0$ requires that $\frac{du}{dx} = 0$.

Hence we have the two equations

$$u = 0, \quad \frac{du}{dx} = 0;$$

by the combination of which, eliminating y, we get a single equation, in terms of x, which will determine those values of x which may or may not render y a maximum or minimum. To decide this, we must pass to $\frac{d^2y}{dx^2}$, which, since $\frac{du}{dx} = 0$, reduces to

$$\frac{d^2y}{dx^2} = -\frac{\frac{d^2u}{dx^2}}{\frac{du}{dy}};$$

and if the values of x and y derived from the equations $u = 0$, $\frac{du}{dx} = 0$, do not cause this to vanish, but make it positive, the value of y corresponding to this value of x is a minimum; but if, by these substitutions for x and y, $\frac{d^2y}{dx^2}$ becomes negative, the value of y is a maximum. But, if these values of x and y cause $\frac{d^2y}{dx^2}$ to vanish, $\frac{d^3y}{dx^3}$ must also vanish in order that y may be a maximum or minimum; and it would be necessary to find $\frac{d^4y}{dx^4}$, and substitute in it, to enable us to decide whether y is a maximum or a minimum.

Ex. 1. Find the value of x that will render y a maximum or minimum in the function

$$u = x^3 - 3axy + y^3 = 0 \quad (1).$$

$$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}} = \frac{ay - x^2}{y^2 - ax}:$$

$$\therefore \qquad \frac{du}{dx} = 0 \text{ answers to } ay - x^2 = 0 \quad (2).$$

From (1) and (2), we find

$$x^6 - 2a^3x^3 = 0:$$

therefore $x = 0$, or $x = a\sqrt[3]{2}$, and the corresponding values of y are $y = 0$, $y = a\sqrt[3]{4}$.

The values $x = 0$, $y = 0$, in the expression for $\frac{dy}{dx}$, cause it to take the form $\frac{0}{0}$; and the true value of $\frac{dy}{dx}$ must be found. This may be done by the method of Art. 113. It is better, however, to proceed as follows: —

The second and third derived equations of the given equation are

$$(y^2 - ax)\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^2 - 2a\frac{dy}{dx} + 2x = 0,$$

$$(y^2 - ax)\frac{d^3y}{dx^3} + \left(6y\frac{dy}{dx} - 3a\right)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 2 = 0;$$

and when, in these, we make $x = 0$, $y = 0$, we find $\frac{dy}{dx} = 0$, and $\frac{d^2y}{dx^2} = \frac{2}{3a}$. Hence $y = 0$ is a minimum when $x = 0$ and $y = 0$. When $x = a\sqrt[3]{2}$, the corresponding value of $y = a\sqrt[3]{4}$ is a maximum; for we have, in this case,

$$\frac{d^2y}{dx^2} = -\frac{\frac{d^2u}{dx^2}}{\frac{du}{dy}} = -\frac{2x}{y^2 - ax} = -\frac{2a\sqrt[3]{2}}{a^2\sqrt[3]{16} - a^2\sqrt[3]{2}} = -\frac{2}{a},$$

which indicates a maximum.

118. Suppose that the relation between x, y, and u, is expressed by the two equations $u = F(x, y)$, $f(x, y) = 0$, so that u is implicitly a function of x; for, deducing the value of y

in terms of x from $f(x, y) = 0$, and substituting this value of y in $u = F(x, y)$, we should have u an explicit function of x. If the maxima and minima values of u were required, we might pursue this course; but we may accomplish our purpose without solving the equation $f(x, y) = 0$.

We have (Art. 82)

$$\frac{du}{dx} = \frac{dF}{dx} + \frac{dF}{dy}\frac{dy}{dx},$$

also (Art. 84)

$$\frac{dy}{dx} = -\frac{\frac{df}{dx}}{\frac{df}{dy}}:$$

hence

$$\frac{du}{dx} = \frac{dF}{dx} - \frac{dF}{dy}\frac{\frac{df}{dx}}{\frac{df}{dy}}.$$

Now, the values of x and y which satisfy simultaneously the equations $f(x, y) = 0$, and $\frac{du}{dx} = 0$, or the equivalent of the latter,

$$\frac{dF}{dx}\frac{df}{dy} - \frac{dF}{dy}\frac{df}{dx} = 0,$$

but which do not cause $\frac{d^2u}{dx^2}$ to vanish, will render u a maximum or a minimum, according to the sign of $\frac{d^2u}{dx^2}$. But, if $\frac{d^2u}{dx^2}$ vanishes for these values of x and y, we must, as before explained, pass on to the derivatives of the higher orders, to enable us to decide the question.

Ex. 1. Given

$$u = x^2 + y^2 = F(x, y) \qquad (1),$$
$$(x - a)^2 + (y - b)^2 - c^2 = 0 = f(x, y) \qquad (2),$$

to find the values of x and y that will make u a maximum or minimum. We find

$$\frac{dF}{dx}=2x,\ \frac{dF}{dy}=2y,\ \frac{df}{dx}=2(x-a),\ \frac{df}{dy}=2(y-b);$$

and therefore

$$\frac{dF}{dx}\frac{df}{dy}-\frac{dF}{dy}\frac{df}{dx}=0$$

becomes

$$x(y-b)-y(x-a)=0:$$

whence $$ay=bx,\ \ y=\frac{b}{a}x.$$

Substituting this value of y in (2), we have

$$x^2-2ax+a^2+\frac{b^2}{a^2}x^2-2\frac{b^2}{a}x+b^2-c^2=0,$$

or $$x^2\left(1+\frac{b^2}{a^2}\right)-2x\left(a+\frac{b^2}{a}\right)+a^2+b^2-c^2=0:$$

whence

$$x=a\pm\frac{ac}{\sqrt{a^2+b^2}}.$$

By differentiation, we get from Eqs. 1 and 2

$$\frac{d^2u}{dx^2}=\mp\frac{2bc^2}{\left(c^2-(x-a)^2\right)^{\frac{3}{2}}}.$$

Observing that the upper sign in the value of x answers to the upper sign in the value of $\frac{d^2u}{dx^2}$, and the lower sign in the one to the lower sign in the other, we discover that

$$x=a+\frac{ac}{\sqrt{a^2+b^2}}$$

renders $\frac{d^2u}{dx^2}$ negative; hence this value of x makes u a maximum: but, when $a-\frac{ac}{\sqrt{a^2+b^2}}$ is substituted for x, $\frac{d^2u}{dx^2}$ be-

comes positive, and u is therefore a minimum for this value of x.

119. When we have n variables connected by $n-1$ equations, by processes of elimination, we may reduce the $n-1$ equations to a single equation involving but two of the variables, and thus bring the investigation of the maxima and minima of the variable which is taken as dependent to the case treated in Art. 117. But, in general, it will be found easier to operate as follows: —

Suppose that the four variables, x, y, z, u, are connected by the three equations,

$$f_1(x, y, z, u) = 0,\ f_2(x, y, z, u) = 0,\ f_3(x, y, z, u) = 0;$$

and that the maximum or minimum value of u is required, x being the independent variable. Differentiating with respect to x, we have

$$\left.\begin{aligned} \frac{df_1}{dx} + \frac{df_1}{dy}\frac{dy}{dx} + \frac{df_1}{dz}\frac{dz}{dx} + \frac{df_1}{du}\frac{du}{dx} = 0 \\ \frac{df_2}{dx} + \frac{df_2}{dy}\frac{dy}{dx} + \frac{df_2}{dz}\frac{dz}{dx} + \frac{df_2}{du}\frac{du}{dx} = 0 \\ \frac{df_3}{dx} + \frac{df_3}{dy}\frac{dy}{dx} + \frac{df_3}{dz}\frac{dz}{dx} + \frac{df_3}{du}\frac{du}{dx} = 0 \end{aligned}\right\} \quad (1).$$

One of the conditions for a maximum or minimum for u being $\frac{du}{dx} = 0$, introducing this in Eqs. 1, they become

$$\left.\begin{aligned} \frac{df_1}{dx} + \frac{df_1}{dy}\frac{dy}{dx} + \frac{df_1}{dz}\frac{dz}{dx} = 0 \\ \frac{df_2}{dx} + \frac{df_2}{dy}\frac{dy}{dx} + \frac{df_2}{dz}\frac{dz}{dx} = 0 \\ \frac{df_3}{dx} + \frac{df_3}{dy}\frac{dy}{dx} + \frac{df_3}{dz}\frac{dz}{dx} = 0 \end{aligned}\right\} \quad (2).$$

The equation which results from the elimination of $\frac{dy}{dx}, \frac{dz}{dx}$

from Eqs. 2, together with the three given equations, which we will denote by $f_1 = 0, f_2 = 0, f_3 = 0$, will determine values of x, y, z, and u. To decide whether any or all of these values, or rather systems of values, make u a maximum or minimum, we must ordinarily pass to $\frac{d^2u}{dx^2}$, and find what sign it takes when the values of the variables are put in it. By differentiating Eqs. 1, the resulting equations and Eqs. 1 will give $\frac{d^2u}{dx^2}$.

120. Before concluding the subject of the maxima and minima of implicit functions, we will briefly refer to the limitations made at the beginning of Art. 117. Resuming the equation

$$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}},$$

we remark, that the necessary condition for a maximum or minimum value of y is, that $\frac{dy}{dx}$ change its sign, which it can do only when it passes through the values 0 or ∞. Now, $\frac{dy}{dx}$ becomes zero when $\frac{du}{dx} = 0$, $\frac{du}{dy}$ being finite; or when $\frac{du}{dy} = \infty$, $\frac{du}{dx}$ being finite. Again: $\frac{dy}{dx}$ becomes infinite when $\frac{du}{dy} = 0$, $\frac{du}{dx}$ remaining finite; or when $\frac{du}{dx} = \infty$, $\frac{du}{dy}$ being finite. It therefore appears that the methods heretofore given for determining the maxima and minima of implicit functions are quite incomplete, as they omit the discussion of several cases that may give rise to these states of value.

Most of the functions with which we have to deal are those

whose maxima and minima are indicated by a change in the sign of the first derivative when it passes through zero. It often happens that the conditions of the problem to be investigated enable us to decide some of the questions relating to maxima and minima, which, if referred to general rules, would require great labor.

EXAMPLES.

1. $u = \frac{1 - x + x^2}{1 + x - x^2}$. $\left\{\begin{array}{l}\text{When } x = \frac{1}{2}, u \text{ is a} \\ \text{minimum.}\end{array}\right.$

2. $u = \frac{x}{1 + x^2}$. $\left\{\begin{array}{l}\text{When } x = 1, u \text{ is a max.} \\ \quad \text{"} \quad x = -1, u \text{ is a min.}\end{array}\right.$

3. $u = e^x + 2\cos. x + e^{-x}$. When $x = 0$, $u = 4$, a min.

4. $u = x^{\frac{1}{x}}$. A max. when $x = e$.

5. Divide the number a into two parts, such that the product of the m^{th} power of one part and the n^{th} power of the other part shall be a maximum.

Ans. The parts are $\frac{na}{m+n}$ and $\frac{ma}{m+n}$; and their product, $m^n n^n \left(\frac{a}{m+n}\right)^{m+n}$ when m and n are even numbers. The product may also have two mimimum states.

6. Find a number such, that, when divided by its Napierian logarithm, the quotient shall be a minimum.

The function to be operated with is $\frac{x}{lx}$.

Ans. $x = e$.

7. $u = \sin. x(1 + \cos. x)$. A max. when $x = \frac{\pi}{3}$.

8. $u = \frac{x}{1 + x \tan. x}$. A max. when $x = \cos. x$.

9. Find the number of equal parts into which a given number a must be divided, that the continued product of these parts may be a maximum.

Ans. $\left\{ \begin{array}{l} \text{Each part must be } e, \text{ the number of} \\ \text{parts } \frac{a}{e}, \text{ and the product } (e)^{\frac{a}{e}}. \end{array} \right.$

10. Of all the triangles standing on a given base, and having equal perimeters, which has the greatest area?

Denote the base by b, and the perimeter by $2p$, and one of the two unknown sides by x.

Ans. $x = \frac{2p - b}{2}$. $\left\{ \begin{array}{l} \text{The triangle} \\ \text{is isosceles.} \end{array} \right.$

11. Of all the squares inscribed in a given square, which is the least?

Ans. That having the vertices of its angles at the middle of the sides of the given square.

12. Inscribe the greatest rectangle in a given semi-ellipse. Let the equation of the ellipse be

$$a^2y^2 + b^2x^2 = a^2b^2.$$

Ans. $\left\{ \begin{array}{l} \text{The sides } a\sqrt{2}, \frac{b}{\sqrt{2}}, \text{ and its} \\ \text{area is } ab. \end{array} \right.$

13. Given the whole surface of a cylinder, required its form when its volume is a maximum.

Represent the whole surface by $2\pi a^2$.

Ans. $\left\{ \begin{array}{l} \text{Radius of the base } \frac{a}{\sqrt{3}}, \text{ axis } \frac{2a}{\sqrt{3}}, \\ \text{volume } \frac{2\pi a^3}{3^{\frac{3}{2}}}. \end{array} \right.$

SECTION XII.

EXPANSION OF FUNCTIONS OF TWO OR MORE INDEPENDENT VARIABLES, AND INVESTIGATION OF THE MAXIMA AND MINIMA OF SUCH FUNCTIONS.

121. LET it be required to develop, and arrange according to the ascending powers of h and k, the function $F(x+h, y+k)$, when $F(x, y)$, and all its partial derivatives up to those of the n^{th} order inclusive, are finite and continuous for all values of x and y included between the values x and $x+h$, y and $y+k$; h and k themselves being finite.

For the time, replace h and k by ht and kt respectively; so that, when it is desired, we may pass back to h and k by making $t=1$. Then $F(x+h, y+k)$ becomes $F(x+ht, y+kt)$. Now, by hypothesis, x, y, h, k, and t, are all independent of each other; and, considered with reference to t alone, we may write

$$F(x+ht, y+kt) = f(t) \quad (1),$$

$$F(x, y) = f(0) \quad (2).$$

For all values of t between the limits $t=0$ and $t=1$, it is evident that $f(t)$ and its derivatives, up to those of the n^{th} order inclusive, satisfy the conditions above imposed on $F(x, y)$ and its derivatives.

Hence, for such values, we have, by Maclaurin's Theorem,

$$f(t) = f(0) + f'(0)\frac{t}{1} + f''(0)\frac{t^2}{1.2} + \cdots$$

$$\cdots + f^{(n-1)}(0)\frac{t^{n-1}}{1.2\ldots(n-1)} + f^{(n)}(\theta t)\frac{t^n}{1.2\ldots n} \quad (3).$$

Deducing the values of $f(0)$, $f'(0)$, $f''(0)\ldots f^{(n)}(t)$, by the method pursued in Art. 110, except that now x and y are not replaced by the particular values a and b, and substituting them and that of $f(t)$ in Eq. 3, we have

$$F(x+ht, y+kt) = F(x,y) + \frac{t}{1}\left(\frac{dF}{dx}h + \frac{dF}{dy}k\right)$$

$$+ \frac{t^2}{1.2}\left(\frac{d^2F}{dx^2}h^2 + 2\frac{d^2F}{dxdy}hk + \frac{d^2F}{dy^2}k^2\right)$$

$$+ \frac{t^3}{1.2.3}\left(\frac{d^3F}{dx^3}h^3 + 3\frac{d^3F}{dx^2dy}h^2k + 3\frac{d^3F}{dxdy^2}hk^2 + \frac{d^3F}{dy^3}k^3\right)$$

$$+ \cdots\cdots\cdots\cdots$$

$$+ \frac{t^n}{1.2\ldots n}\left(\frac{d^nF}{dx^n}h^n + n\frac{d^nF}{dx^{n-1}dy}h^{n-1}k + \cdots + \frac{d^nF}{dy^n}\right)_{\substack{x=x+\theta ht \\ y=y+\theta kt}} \quad (4).$$

The notations $x = x + \theta ht$, $y = y + \theta kt$, attached to the parenthesis of the last term, signify that in the derivatives $\frac{d^nF}{dx^n}, \frac{d^nF}{dx^{n-1}dy}, \ldots \frac{d^nF}{dy^n}$, x is replaced by $x + \theta ht$, and y by $y + \theta kt$.

In (4), make $t = 1$, and we have

$$F(x+h, y+k) = F(x,y) + \frac{dF}{dx}h + \frac{dF}{dy}k$$

$$+ \frac{1}{1.2}\left(\frac{d^2F}{dx^2}h^2 + 2\frac{d^2F}{dxdy}hk + \frac{d^2F}{dy^2}k^2\right)$$

$$+ \frac{1}{1.2.3}\left(\frac{d^3F}{dx^3}h^3 + 3\frac{d^3F}{dx^2dy}h^2k + 3\frac{d^3F}{dxdy^2}hk^2 + \frac{d^3F}{dy^3}k^3\right)$$

$$+ \cdots\cdots\cdots\cdots$$

$$+ \frac{1}{1.2\ldots n}\left(\frac{d^nF}{dx^n}h^n + n\frac{d^nF}{dx^{n-1}dy}h^{n-1}k + \cdots + \frac{d^nF}{dy^n}k^n\right)_{\substack{x=x+\theta h \\ y=y+\theta k}} \quad (5);$$

which is the development sought.

If, in Eq. 5, we make $x = 0$ and $y = 0$, and then in the result write x for h, and y for k, we find

$$F(x, y) = F(0, 0) + \left(\frac{dF}{dx}\right)_{\substack{x=0\\y=0}} x + \left(\frac{dF}{dy}\right)_{\substack{x=0\\y=0}} y$$

$$+ \frac{1}{1.2}\left\{\left(\frac{d^2F}{dx^2}\right)_{\substack{x=0\\y=0}} x^2 + 2\left(\frac{d^2F}{dxdy}\right)_{\substack{x=0\\y=0}} xy + \left(\frac{d^2F}{dy^2}\right)_{\substack{x=0\\y=0}} y^2\right\}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{1}{1.2.3\ldots n}\left\{\left(\frac{d^nF}{dx^n}\right)_{\substack{x=\theta x\\y=\theta y}} x^n + n\left(\frac{d^nF}{dx^{n-1}dy}\right)_{\substack{x=\theta x\\y=\theta y}} x^{n-1}y + \cdots \right.$$

$$\left. + \left(\frac{d^nF}{dy^n}\right)_{\substack{x=\theta x\\y=\theta y}} y^n\right\} \quad (6);$$

which is the formula for the development of a function of two independent variables into a series arranged according to the ascending powers of the variables.

The extension of formulas (5) and (6) of this article to functions of more than two variables is easily made. For the expansion of $F(x + h,\ y + k,\ z + i \ldots)$, we should find

$$F(x+h,\ y+k,\ z+i\ldots) = F(x, y, z\ldots),$$

$$+ \frac{dF}{dx}h + \frac{dF}{dy}k + \frac{dF}{dz}i + \cdots$$

$$+ \frac{1}{1.2}\left(\frac{d^2F}{dx^2}h^2 + \frac{d^2F}{dy^2}k^2 + \cdots + 2\frac{d^2F}{dxdy}hk + \cdots\right)$$

$$+ \frac{1}{1.2.3}\left(\frac{d^3F}{dx^3}h^3 + \cdots + 3\frac{d^3F}{dx^2dy}h^2k + \cdots + 3\frac{d^3F}{dxdy^2}hk^2 + \cdots\right)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \frac{1}{1.2\ldots n}\left(\frac{d^nF}{dx^n}h^n + \frac{d^nF}{dy^n}k^n + \cdots \right.$$

$$\left. + n\frac{d^nF}{dx^{n-1}dy}h^{n-1}k + \cdots\right)_{\substack{x=x+\theta h\\y=y+\theta k\\z=z+\theta i\\ \vdots\ \ \vdots\ \ \vdots\ \ \vdots}} \quad (7);$$

and if, in this, we first make $x, y, z \ldots$, severally equal to zero, and then in the result write $x, y, z \ldots$, for $h, k, i \ldots$, respectively, we have

$$F(x,y,z\ldots) = F(x,y,z\ldots)_0 + \left(\frac{dF}{dx}\right)_0 x + \left(\frac{dF}{dy}\right)_0 y + \left(\frac{dF}{dz}\right)_0 z + \cdots$$

$$+ \frac{1}{1.2}\left\{\left(\frac{d^2F}{dx^2}\right)_0 x^2 + \left(\frac{d^2F}{dy^2}\right)_0 y^2 + \left(\frac{d^2F}{dz^2}\right)_0 z^2 + \cdots \right.$$
$$\left. + 2\left(\frac{d^2F}{dxdy}\right)_0 xy + \cdots \right\}$$

$$+ \frac{1}{1.2.3}\left\{\left(\frac{d^3F}{dx^3}\right)_0 x^3 + \cdots + \left(\frac{d^3F}{dx^2dy}\right)_0 x^2y + \cdots \right.$$
$$\left. + 3\left(\frac{d^3F}{dxdy^2}\right)_0 xy^2 + \cdots \right\}$$

$$+ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

$$+ \frac{1}{1.2\ldots n}\left\{\left(\frac{d^nF}{dx^n}\right)x^n + \left(\frac{d^nF}{dy^n}\right)y^n + \cdots \right.$$
$$\left. + n\left(\frac{d^nF}{dx^{n-1}dy}\right)x^{n-1}y + \cdots \right\}_{\substack{x=\theta x \\ y=\theta y \\ z=\theta z \\ \vdots}} \quad (8);$$

a formula for the development of a function of any number of independent variables, and in which the notation $(\;)_0$ signifies that the variables entering the expression within the parentheses are made zero. In formulas from (5) to (8) inclusive, of this article, the last terms are remainders expressing the difference between the value of the sum of the preceding terms of the development and the value of the function. When the form of the function under consideration, and the values attributed to the variables, are such, that, as n increases without

limit, the remainder decreases without limit, then, by taking n sufficiently great, the remainders may be neglected.

122. Maxima and minima of functions of two or more independent variables.

A function $F(x, y, z \ldots)$ of several independent variables is a maximum, when, being real, it is, for certain values of the variables, greater than $F(x+h,\ y+k,\ z+i \ldots)$; the increments $h, k, i \ldots$, being very small, and taken with all possible combinations of signs. On the contrary, the function is a minimum, when, under the same conditions, it is less than $F(x+h,\ y+k,\ z+i \ldots)$. Let us consider first the function $F(x, y)$ of the two independent variables x and y, and endeavor to deduce from the conditions of these definitions, the criteria of a maximum or minimum of this function. Resuming Eq. 5, Art. 121, stopping in the second member at those terms which involve the third order of the partial derivatives of the function, and transposing $F(x, y)$ to the first member, we have

$$
\begin{aligned}
F(x+h, y+k) - F(x, y) &= \frac{dF}{dx}h + \frac{dF}{dy}k \\
&+ \frac{1}{1.2}\left(\frac{d^2F}{dx^2}h^2 + 2\frac{d^2F}{dxdy}hk + \frac{d^2F}{dy^2}k^2\right) \\
&+ \frac{1}{1.2.3}\left(\frac{d^3F}{dx^3}h^3 + 3\frac{d^3F}{dx^2dy}h^2k + 3\frac{d^3F}{dxdy^2}hk^2 + \frac{d^3F}{dy^3}k^3\right)_{\substack{x=x+\theta h \\ y=y+\theta k}} \quad (1),
\end{aligned}
$$

the last term of which we will denote by R.

Now, if $F(x, y)$ is a maximum, the first member of (1) is negative; and therefore its second member must be negative also, and this whether h and k be both positive or both negative, or either one be positive and the other negative; and whatever be the values of h and k, provided only that they be very small. If $F(x, y)$ is a minimum, the second member of

(1) must be positive under the same conditions and limitations in respect to the signs and values of h and k.

But, when h and k are taken sufficiently small, the sign of the second member of (1) will be the same as that of $\frac{dF}{dx}h + \frac{dF}{dy}k$, which must therefore be permanent and negative if $F(x, y)$ is a maximum, and permanent and positive if $F(x, y)$ is a minimum. It is plain, however, that the sign of $\frac{dF}{dx}h + \frac{dF}{dy}k$ will change by changing the signs of h and k. To make the sign of the second member of (1) invariable, whether positive or negative, we must have

$$\frac{dF}{dx}h + \frac{dF}{dy}k = 0;$$

and, since h and k are entirely independent of each other, this requires that

$$\frac{dF}{dx} = 0, \text{ and } \frac{dF}{dy} = 0 \quad (2).$$

Let $x = a$, $y = b$, be values of x and y derived from these equations, and denote by A, B, C, R_1, what $\frac{d^2F}{dx^2}, \frac{d^2F}{dxdy}, \frac{d^2F}{dy^2}, R$, respectively become when these values of x and y are substituted in (1); then (1) becomes

$$F(a+h, b+k) - F(a, b) = \frac{1}{1.2}(Ah^2 + 2Bhk + Ck^2) + R_1 \quad (3).$$

When the values of h and k are very small, and only such values are admissible, the sign of the second member of (3) will be the same as that of

$$Ah^2 + 2Bhk + Ck^2,$$

which may be put under the form

$$Ak^2\left(\frac{h^2}{k^2} + 2\frac{B}{A}\frac{h}{k} + \frac{C}{A}\right).$$

The sign of this will be invariable, and the same as that of A, if the roots of the equation

$$\frac{h^2}{k^2} + 2\frac{B}{A}\frac{h}{k} + \frac{C}{A} = 0 \quad (4)$$

are imaginary; $\frac{h}{k}$ being treated as the unknown quantity.

Solving this equation, we find

$$\frac{h}{k} = \frac{-B \pm \sqrt{B^2 - AC}}{A};$$

from which we conclude that the conditions for imaginary roots are, that A and C have the same sign, and that the product AC be greater than B^2.

In recapitulation, we say, that if $x = a$, $y = b$, make $F(x, y)$ a maximum then for these values of x and y we must have

$$\frac{dF}{dx} = 0, \frac{dF}{dy} = 0,$$

$$\frac{d^2F}{dx^2}, \frac{d^2F}{dy^2}, \text{ both negative,}$$

and

$$\frac{d^2F}{dx^2}\frac{d^2F}{dy^2} > \left(\frac{d^2F}{dxdy}\right)^2.$$

If $x = a$, $y = b$, make $F(x, y)$ a minimum, the conditions are the same, except that then $\frac{d^2F}{dx^2}, \frac{d^2F}{dy^2}$, must both be positive.

The existence of real roots for Eq. 4 indicates that we may give such signs and values to h and k as to cause the expression $Ah^2 + 2Bhk + Ck^2$ to vanish, and, in so doing, change its sign, which is incompatible with the existence of a maximum or a minimum state of $F(x, y)$.

123. There remains to be examined the case in which $AC - B^2 = 0$. When this condition presents itself, there may also be a maximum or minimum value of the function.

By the theory of the composition of equations, or by inspection, the expression $Ah^2 + 2Bhk + Ck^2$ may be written

$$\frac{k^2}{A}\left\{\left(A\frac{h}{k}+B\right)^2 + AC - B^2\right\}:$$

hence, when $AC - B^2 = 0$, this becomes $\frac{k^2}{A}\left(A\frac{h}{k}+B\right)$, the sign of which, except when $\left(A\frac{h}{k}+B\right)^2$ vanishes, is the same as that of A; and $F(a, b)$ is a maximum if A is a negative, and a minimum if A is positive. Should $\left(A\frac{h}{k}+B\right)^2$ vanish, as it does when $\frac{h}{k} = -\frac{B}{A}$, we cannot tell, without further inquiry, that $F(a+h, b+k) - F(a, b)$ does not undergo a change of sign. To decide this, let L, M, N, P, represent the values of $\frac{d^3F}{dx^3}, \frac{d^3F}{dx^2dy}, \frac{d^3F}{dxdy^2}, \frac{d^3F}{dy^3}$, respectively, when $x = a$, $y = b$; and also put R_2 for the value of

$$\left(\frac{d^4F}{dx^4}h^4 + 4\frac{d^4F}{dx^3dy}h^3k + \cdots + 4\frac{d^4F}{dxdy^3}hk^3 + \frac{d^4F}{dy^4}k^4\right)_{\substack{x=x+\theta h \\ y=y+\theta k}}$$

for the same values of x and y.

Introducing these values, and the conditions

$$\frac{dF}{dx} = 0, \ \frac{dF}{dy} = 0, \ Ah^2 + 2Bhk + Ck = 0,$$

the latter being a consequence of the hypotheses $\frac{h}{k} = -\frac{B}{A}$ and $AC - B^2 = 0$, we have

$F(a+h, b+k) - F(a, b) =$

$$\frac{1}{1.2.3}(Lh^3 + 3Mh^2k + 3Nhk^2 + Pk^3) + R_2$$

From this we see, that since the sign of $F(a+h, b+k) - F(a, b)$, when h and k are very small, is the same as that of

$$Lh^3 + 3Mh^2k + 3Nhk^2 + Pk^3,$$

$F(a, b)$ cannot be a maximum or minimum, unless, if $A\frac{h}{k} + B$ vanishes,

$$Lh^3 + 3Mh^2k + 3Nhk^2 + Pk^3$$

vanishes simultaneously. Suppose these conditions to be satisfied, then the sign of $F(a+h, b+k) - F(a, b)$ is the same as that of R_2. But we have shown, that when $A\frac{h}{k} + B$ was not equal to zero, and h and k were taken sufficiently small, the sign of $F(a+h, b+k) - F(a, b)$ is the same as that of A; but, when $F(a, b)$ is a maximum or minimum, the sign of $F(a+h, b+k) - F(a, b)$ must be invariable for all values of h and k which are small enough to cause $F(x, y)$ to change in value by continuous degrees in the immediate vicinity of the value $F(a, b)$. Hence it follows, that, when the values of h and k are such as to make $\frac{h}{k} = -\frac{B}{A}$, these values must give R_2 a sign the same as that of A. If these several conditions are satisfied, the function is a maximum when A is negative, and a minimum when A is positive.

124. If $A = 0$, $B = 0$, $C = 0$, then

$$F(a+h, b+k) - F(a, b) = \frac{1}{1.2.3}(Lh^3 + 3Mh^2k + 3Nhk^2 + Pk^3) + R_2;$$

L, M, N, P, R_2, denoting the same values as in the preceding article. Hence, in order that $F(a, b)$ may be a maximum or minimum, L, M, N, P, must separately vanish, and the sign of R_2 must be invariable; and generally, when $F(a, b)$ is a maximum or minimum, all the partial derivatives up to those of an

odd order inclusive must vanish; while those of the following even order are so related, that the sign of the expression in which they are the co-efficients of the powers and products of h and k remains the same, whatever be the signs of h and k. The conditions which will insure this invariability of sign when the derivatives are of a higher order than the second, are, in the general case, too complicated to be here discussed, or even to be of much practical value.

125. Let it now be required to find the maxima and minima values of $F(x, y, z)$, a function of the three independent variables x, y, z.

Referring to Eq. 7, Art. 121, and in the second member stopping with the terms involving the partial derivatives of the third order, denoting the aggregate of the remaining terms by R, and carrying $F(x, y, z)$ to the first member, we have

$$F(x+h, y+k, z+i) - F(x, y, z) = \frac{dF}{dx}h + \frac{dF}{dy}k + \frac{dF}{dz}i$$
$$+ \frac{1}{1.2}\left(\frac{d^2F}{dx^2}h^2 + \frac{d^2F}{dy^2}k^2 + \frac{d^2F}{dz^2}i^2 + 2\frac{d^2F}{dxdy}hk + 2\frac{d^2F}{dxdz}hi\right.$$
$$\left. + 2\frac{d^2F}{dydz}ki\right) + R \quad (1).$$

When h, k, i, have the very small values which alone are admissible in our investigation, the sign of the second member of (1) will depend upon that of the expression

$$\frac{dF}{dx}h + \frac{dF}{dy}k + \frac{dF}{dz}i;$$

and the sign of $F(x+h, y+k, z+i) - F(x, y, z)$ cannot remain invariable for all the possible values and combinations of the signs of h, k, i, unless

$$\frac{dF}{dx}h + \frac{dF}{dy}k + \frac{dF}{dz}i = 0;$$

which is, therefore, the first condition necessary to insure a maximum or minimum of $F(x, y, z)$. But, because h, k, i, are entirely independent of each other, the above condition involves the three following: —

$$\frac{dF}{dx} = 0, \frac{dF}{dy} = 0, \frac{dF}{dz} = 0 \quad (2),$$

which will determine one or more systems of values for x, y, z. And we now proceed to inquire what further conditions must be satisfied when one of these systems, say $x = a$, $y = b$, $z = c$, renders the function a maximum or minimum.

Substituting these values in (1), and representing the values which $\frac{d^2F}{dx^2}, \frac{d^2F}{dy^2}, \frac{d^2F}{dz^2}, \frac{d^2F}{dxdy}, \frac{d^2F}{dxdz}, \frac{d^2F}{dydz}, R$, then assume, by A, B, C, A', B', C', R_1, respectively, we have

$$F(a + h, b + k, c + i) - F(a, b, c) =$$
$$\frac{1}{1.2}(Ah^2 + Bk^2 + Ci^2 + 2A'hk + 2B'hi + 2C'ki) + R_1 \quad (3),$$

the sign of the second member of which, when h, k, i, have very small values, is the same as that of the expression

$$Ah^2 + Bk^2 + Ci^2 + 2A'hk + 2B'hi + 2C'ki \quad (a);$$

and this sign must be permanent during all the changes in the signs and values of h, k, i, if $F(a, b, c)$ is a maximum or minimum of $F(x, y, z)$.

Expression (a) may be written

$$i^2\left(A\frac{h^2}{i^2} + B\frac{k^2}{i^2} + C + 2A'\frac{h}{i}\frac{k}{i} + 2B'\frac{h}{i} + 2C'\frac{k}{i}\right).$$

Make $s = \frac{h}{i}$, $t = \frac{k}{i}$; then we have

$$i^2(As^2 + Bt^2 + C + 2A'st + 2B's + 2C't);$$

and the sign of this last will be the same as that of

$$As^2 + Bt^2 + C + 2A'st + 2B's + 2C't,$$

or $$A\left(s^2 + \frac{B}{A}t^2 + \frac{C}{A} + 2\frac{A'}{A}st + 2\frac{B'}{A}s + 2\frac{C'}{A}t\right) \quad (b).$$

The conditions that will make the sign of (b) invariable for all possible values of s and t will make that of (a) invariable for all the combinations of signs, and all admissible values of h, k, i. To find these conditions, put (b) equal to zero, and solve the resulting equation with respect to either s or t, say s. We thus get

$$s = -\frac{A't + B'}{A} \pm \frac{1}{A}\sqrt{\left\{(A'^2 - AB)t^2 + 2(A'B' - AC')t + B'^2 - AC\right\}}.$$

Now, if the quantities A, B, C, A', B', C', have such relative values, that the quantity under the radical, in this value of s, cannot become positive for any real value of t, then the parenthetical factor of (b) will always be positive, and the sign of (b) will be the same as that of A. Putting this quantity under the form

$$(A'^2 - AB)\left(t^2 + 2\frac{A'B' - AC'}{A'^2 - AB}t + \frac{B'^2 - AC}{A'^2 - AB}\right),$$

we see, that, to make it negative for all values of t, it is necessary and sufficient to have

$$A'^2 - AB < 0, \text{ i.e., } AB - A'^2 > 0 \qquad (c).$$

$$(A'B' - AC')^2 < (B'^2 - AC)(A'^2 - AB) \quad (d).$$

When conditions (c) and (d) are satisfied by the values of x, y, z, deduced from Eqs. 2, $F(a, b, c)$ is a maximum or a minimum; for then the sign of expression (b), and consequently that of the second member of Eq. 3, is permanent, and the same as that of A. $F(a, b, c)$ is therefore a maximum, if A is negative; and a minimum, if A is positive.

Hence the conditions necessary for the existence of a maximum or minimum of $F(x, y, z)$ are, that the values of x, y, z, derived from the equations

$$\frac{dF}{dx} = 0, \frac{dF}{dy} = 0, \frac{dF}{dz} = 0,$$

should make

$$\frac{d^2F}{dx^2}\frac{d^2F}{dy^2} - \left(\frac{d^2F}{dxdy}\right)^2 > 0, \text{ that is, positive} \quad (c');$$

and

$$\left(\frac{d^2F}{dxdy}\frac{d^2F}{dxdz} - \frac{d^2F}{dx^2}\frac{d^2F}{dydz}\right)^2 <$$

$$\left\{\left(\frac{d^2F}{dxdz}\right)^2 - \frac{d^2F}{dx^2}\frac{d^2F}{dz^2}\right\}\left\{\left(\frac{d^2F}{dxdy}\right)^2 - \frac{d^2F}{dx^2}\frac{d^2F}{dy^2}\right\} \quad (d').$$

A necessary consequence of conditions (c') and (d') is, that

$$\left(\frac{d^2F}{dxdz}\right)^2 - \frac{d^2F}{dx^2}\frac{d^2F}{dz^2} < 0, \text{ or } \frac{d^2F}{dx^2}\frac{d^2F}{dz^2} - \left(\frac{d^2F}{dxdz}\right)^2 > 0;$$

and hence $\frac{d^2F}{dx^2}, \frac{d^2F}{dy^2}, \frac{d^2F}{dz^2}$, must all have the same sign, which is negative when $F(x, y, z)$ is a maximum, and positive when $F(x, y, z)$ is a minimum.

126. If we have a function $F(x, y, z \dots)$ of n independent variables, the first condition for the existence of a maximum or minimum would be

$$\frac{dF}{dx}h + \frac{dF}{dy}k + \frac{dF}{dz}i + \cdots = 0:$$

whence, because $h, k, i \dots$, are independent of each other,

$$\frac{dF}{dx} = 0, \ \frac{dF}{dy} = 0, \ \frac{dF}{dz} = 0 \quad (1).$$

Eqs. 1 determine values $x = a$, $y = b$, $z = c \dots$, which may or may not produce a maximum or minimum state of the func-

tion. To decide this question, we should have to examine the term

$$\frac{1}{1.2}\left(\frac{d^2F}{dx^2}h^2 + \frac{d^2F}{dy^2}k^2 + \frac{d^2F}{dz^2}i^2 + \cdots + 2\frac{d^2F}{dxdy}hk + \cdots\right)$$

in the expression for

$$F(x+h,\ y+k,\ z+i\ldots) - F(x,\ y,\ z\ldots).$$

If, for these values of $x, y, z\ldots$, the sign of this term is permanent, and negative for all admissible values, and all the combinations of the signs of $h, k, i\ldots$, the function is a maximum: if the sign is permanent and positive, the function is a minimum. It would be found, that, to insure either of these states of the function, $\frac{d^2F}{dx^2}, \frac{d^2F}{dy^2}, \frac{d^2F}{dz^2}\cdots$, must all have the same sign, negative for a maximum, positive for a minimum. But the investigation of all the conditions to be satisfied in this general case, in order that the function may be a maximum or minimum, is too complicated to find a place in an elementary work.

127. Maxima and minima of a function of several variables some of which are dependent on the others.

Let it be required to find the maxima and minima values of the function $u = F(x, y, z\ldots)$ of the m variables $x, y, z\ldots$, which are connected by the n equations

$$\left.\begin{array}{l} f_1(x, y, z\ldots) = 0 \\ f_2(x, y, z\ldots) = 0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ f_n(x, y, z\ldots) = 0 \end{array}\right\} \quad (1).$$

By eliminating from u, n of the m variables, by means of the n given equations, u would become an explicit function of $m-n$ independent variables, and its maximum or minimum could

then be found by the method just explained; but this elimination may be avoided, and the determination of the maxima and minima of u greatly simplified, by the process which follows: —

Suppose the variables $x, y, z \ldots$, to receive the respective increments $h, k, i \ldots$, by virtue of which the function passes from a given state of value to another in the immediate vicinity of this: then, if the given state be either a maximum or a minimum, we must have

$$\frac{dF}{dx}h + \frac{dF}{dy}k + \frac{dF}{dz}i + \cdots = 0 \quad (2).$$

The partial derivatives of the first members of each of Eqs. 1, taken with respect to each of the variables, are separately equal to zero. Taking these equations in succession, multiplying each partial derivative by the increment of that variable to which the derivative relates, and placing the sum of the results equal to zero, we have

$$\left.\begin{array}{l} \frac{df_1}{dx}h + \frac{df_1}{dy}k + \frac{df_1}{dz}i + \cdots = 0 \\ \frac{df_2}{dx}h + \frac{df_2}{dy}k + \frac{df_2}{dz}i + \cdots = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \frac{df_n}{dx}h + \frac{df_n}{dy}k + \frac{df_n}{dz}i + \cdots = 0 \end{array}\right\} \quad (3).$$

There being a number (n) of Eqs. 3, these with (2) make $n + 1$ equations of the first degree in respect to the m quantities $h, k, i \ldots$: hence, by the combination of these equations, we may eliminate n of these quantities, and arrive at a final equation involving the remaining $m - n$ quantities, and also of the first degree with respect to them. To facilitate this elimination, let $\mu_1, \mu_2 \ldots, \mu_n$, denote undetermined quantities; and

multiply the first of Eqs. 3 by μ_1, the second by $\mu_2 \ldots$, the n^{th} by μ_n; add the results to (2), and arrange with reference to $h, k, i \ldots$: we thus get

$$\left.\begin{array}{l}
\left(\frac{dF}{dx} + \mu_1 \frac{df_1}{dx} + \mu_2 \frac{df_2}{dx} + \cdots + \mu_n \frac{df_n}{dx}\right) h \\
+ \left(\frac{dF}{dy} + \mu_1 \frac{df_1}{dy} + \mu_2 \frac{df_2}{dy} + \cdots + \mu_n \frac{df_n}{dx}\right) k \\
+ \left(\frac{dF}{dz} + \mu_1 \frac{df_1}{dz} + \mu_2 \frac{df_2}{dz} + \cdots + \mu_n \frac{df_n}{dz}\right) i \\
+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}\right\} = 0 \quad (4);$$

a true equation when $F(x, y, z \ldots)$ is susceptible of a maximum or minimum, whatever be the values of $\mu_1, \mu_2 \ldots, \mu_n$.

Place the co-efficients of n of the quantities $h, k, i \ldots$, in Eq. 4, equal to zero: the n equations thus obtained will determine $\mu_1, \mu_2 \ldots, \mu_n$. By substituting these values of $\mu_1, \mu_2 \ldots, \mu_n$, in (4), n of the quantities $h, k, i \ldots$, will vanish from that equation; and, if the co-efficients of the $m - n$ of these quantities remaining in the equation be placed equal to zero, we have, including the n given equations,

$$f_1(x, y, z \ldots) = 0 \ldots f_n(x, y, z \ldots) = 0,$$

m equations from which to determine values $a, b, c \ldots$, for the m quantities $x, y, z \ldots$, respectively. This is equivalent to equating the co-efficient of each of the m quantities $h, k, i \ldots$, in (4), to zero; and these m equations, together with the n given equations, will make $m + n$ equations, by means of which we may eliminate the n indeterminates $\mu_1, \mu_2 \ldots, \mu_n$, and find the m quantities $x, y, z \ldots$

It remains to be ascertained whether the sign of the expression for $F(a + h, b + k, c + i \ldots) - F(a, b, c \ldots)$ is invariable; and, if so, whether it be positive, which answers to a minimum;

or negative, which answers to a maximum. Theoretically, this examination is very complicated; but, for most cases in which this method is applicable, the form of the function enables us to decide at once which of the two states, if either, the function admits.

When $m - n = 1$, or there is only one more variable than there are equations connecting them, the case discussed in this article reduces to that of an expression which is implicitly a function of a single variable.

128. In the case in which it is required to determine the maxima or minima of a function, the several variables of which are connected by but one equation, the process may be still further simplified.

Let $u = F(x, y, z \ldots)$ be the function, and

$$f(x, y, z \ldots) = 0 \quad (1),$$

the equation expressing the relation between the variables $x, y, z \ldots$: then, by the reasoning employed in the last article, we have

$$\frac{dF}{dx}h + \frac{dF}{dy}k + \frac{dF}{dz}i + \cdots = 0 \quad (2),$$

$$\frac{df}{dx}h + \frac{df}{dy}k + \frac{df}{dz}i + \cdots = 0 \quad (3).$$

Multiplying (3) by the indeterminate μ, subtracting the result from (2), and arranging with reference to $h, k, i \ldots$, we have

$$\left(\frac{dF}{dx} - \mu\frac{df}{dx}\right)h + \left(\frac{dF}{dy} - \mu\frac{df}{dy}\right)k + \left(\frac{dF}{dz} - \mu\frac{df}{dz}\right)i + \cdots = 0 \quad (4).$$

Equating to zero the co-efficients of the several quantities $h, k, i \ldots$, we should have, with the given equation, $m + 1$ equations, by means of which we can eliminate μ, and determine the m quantities. But from the co-efficients of $h, k, i \ldots$, in (4), placed equal to zero, we find

$$\mu = \frac{\frac{dF}{dx}}{\frac{df}{dx}} = \frac{\frac{dF}{dy}}{\frac{df}{dy}} = \frac{\frac{dF}{dz}}{\frac{df}{dz}} = \cdots;$$

that is, the ratio of the co-efficients of h, in Eqs. 2 and 3, is the same as that of the co-efficients of k, of $i\ldots$ This relation will be found to facilitate the determination of maxima and minima.

The examples which follow are arranged in the order of the articles in this section under which they fall.

EXAMPLES.

1. If $F(x, y) = x^2(a + y)^3$, find the expansion of

$$(x + h)^2(a + y + k)^3$$

in the ascending powers of h and k.

$$(x+h)^2(a+y+k)^3 = \begin{cases} x^2(a+y)^3 + 2x(a+y)^3h + 3x^2(a+y)^2k \\ +(a+y)^3h + 6x(a+y)^2hk + 3x^2(a+y)k^2 \\ +3(a+y)^2h^2k + 6x(a+y)hk^2 + x^2k^3 \\ +3(a+y)h^2k^2 + 2xhk^3 + h^2k^3. \end{cases}$$

2. If $F(x, y, z) = ax^2 + by^2 + cz^2 + 2exy + 2gxz + 2fyz$, find the expansion of $F(x + h, y + k, z + i)$.

$$F(x+h, y+k, z+i) = \begin{cases} ax^2 + by^2 + cz^2 + 2exy + 2gxz + 2fyz \\ +2(ax + gz + ey)h + 2(by + fz + ex)k \\ +2(cz + fy + gx)i + (ah^2 + bk^2 + ci^2) \\ +2(fki + ghi + ehk). \end{cases}$$

3. Expand

$$\left(a_0 + a_1\frac{x}{1} + a_2\frac{x^2}{1.2} + \cdots\right)\left(b_0 + b_1\frac{y}{1} + b_2\frac{y^2}{1.2} + \cdots\right)$$

in the ascending powers of x and y.

$$\left(a_0 + a_1\frac{x}{1} + a_2\frac{x^2}{1.2} + \cdots\right)\left(b_0 + b_1\frac{y}{1} + b_2\frac{y^2}{1.2} + \cdots\right)$$
$$= a_0b_0 + a_1b_0x + a_0b_1y$$
$$+ \frac{1}{1.2}(a_2b_0x^2 + 2a_1b_1xy + a_0b_2y^2)$$
$$+ \frac{1}{1.2.3}(a_3b_0x^3 + 3a_2b_1x^2y + 3a_1b_2xy^2 + a_0b_3y^3)$$
$$+ \cdots\cdots\cdots\cdots$$

4. Find what values, if any, of x and y, will render the function $F(x, y) = x^2y + xy^2 - axy$ a maximum or minimum.

From the equations $\frac{dF}{dx} = 0$, $\frac{dF}{dy} = 0$, we get four systems of values, viz.

$$\left.\begin{matrix} x = 0 \\ y = 0 \end{matrix}\right\},\quad \left.\begin{matrix} x = a \\ y = 0 \end{matrix}\right\},\quad \left.\begin{matrix} x = 0 \\ y = a \end{matrix}\right\},\quad \left.\begin{matrix} x = \frac{a}{3} \\ y = \frac{a}{3} \end{matrix}\right\};$$

none of the first three of which satisfy the condition

$$\frac{d^2F}{dx^2}\,\frac{d^2F}{dy^2} > \left(\frac{d^2F}{dxdy}\right)^2 \quad \text{(Art. 123)},$$

and must therefore be rejected. The fourth system reduces this inequality to $\frac{4}{9}a^2 > \frac{a^2}{9}$, which is true, and at the same time makes both $\frac{d^2F}{dx^2}$ and $\frac{d^2F}{dy^2}$ positive: hence the values $x = \frac{a}{3}$, $y = \frac{a}{3}$, make the function a minimum, and this minimum is $-\frac{a^3}{27}$.

5. Determine the values of x and y that will make

$$F(x, y) = e^{-(x^2+y^2)}(ax^2 + by^2)$$

a maximum or minimum.

$\frac{dF}{dx} = 0$, $\frac{dF}{dy} = 0$, give the three systems of values

$$\left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\}, \quad \left.\begin{array}{l} x = 0 \\ y = \pm 1 \end{array}\right\}, \quad \left.\begin{array}{l} x = \pm 1 \\ y = 0 \end{array}\right\},$$

which we will examine in succession.

The first system gives $\frac{d^2F}{dx^2} = 2a$, $\frac{d^2F}{dy^2} = 2b$, $\frac{d^2F}{dxdy} = 0$: hence the values $x = 0$, $y = 0$, make the function a minimum. With the second system, we have

$$\frac{d^2F}{dx^2} = 2(a - b)e^{-1}, \quad \frac{d^2F}{dy^2} = -4be^{-1}, \quad \frac{d^2F}{dxdy} = 0:$$

hence the existence of a maximum or minimum depends on the relative values of a and b. If b is greater than a, $\frac{d^2F}{dx^2}$, $\frac{d^2F}{dy^2}$, have the same sign, which is negative, and the function is a maximum; but, if b be less than a, $\frac{d^2F}{dx^2}$, $\frac{d^2F}{dy^2}$, have opposite signs, and the second system of values of x and y make the function neither a maximum nor a minimum.

For the third system, we find

$$\frac{d^2F}{dx^2} = -4ae^{-1}, \quad \frac{d^2F}{dy^2} = 2(b - a)e^{-1}, \quad \frac{d^2F}{dxdy} = 0:$$

from which we conclude that $x = \pm 1$, $y = 0$, will make the function a maximum when $a > b$; but, when $a < b$, it has neither a maximum nor a minimum.

6. The equations of two planes, referred to rectangular co-ordinate axes, are

$$f_1(x, y, z) = Ax + By + Cz - D = 0 \quad (1),$$
$$f_2(x, y, z) = A'x + B'y + C'z - D' = 0 \quad (2).$$

It is required to find the shortest distance from the origin of co-ordinates to the line of intersection of the planes.

Let $$F(x, y, z) = x^2 + y^2 + z^2 \quad (3)$$
represent the square of the distance from the origin to the

point of which x, y, z, are the co-ordinates: then, if x, y, z, are the same in the three Eqs. 1, 2, 3, the concrete question is reduced to the abstract one of finding the values of x, y, z; which, when subject to the conditions of Eqs. 2 and 3, will render $F(x, y, z)$ a minimum.

By Art. 128, we have

$$\left.\begin{array}{l} 2x + \mu_1 A + \mu_2 A' = 0 \\ 2y + \mu_1 B + \mu_2 B' = 0 \\ 2z + \mu_1 C + \mu_2 C' = 0 \end{array}\right\} \quad (4).$$

Multiplying the first of Eqs. 4 by A, the second by B, and the third by C, adding the results, then, by (1), we have

$$(A^2 + B^2 + C^2)\mu_1 + (AA' + BB' + CC')\mu_2 + 2D = 0 \quad (5).$$

In like manner,

$$(A'^2 + B'^2 + C'^2)\mu_1 + (AA' + BB' + CC')\mu_2 + 2D' = 0 \quad (6).$$

From Eqs. 5 and 6, we get the values of μ_1, μ_2; and Eqs. 4, when these values of μ_1, μ_2, are substituted in them, will determine x, y, z. Multiplying the first of (4) by x, the second by y, and the third by z, adding results, and reducing by Eqs. 1, 2, and 3, we have

$$2F(x, y, z) + D\mu_1 + D'\mu_2 = 0;$$

from which we get $F(x, y, z)$. In this case, it is unnecessary to examine the sign of $F(x+h,\ y+k,\ c+i) - F(x, y, z)$, when the values of x, y, z, are substituted; for we know from the conditions of the geometrical question that the function has a minimum.

7. Required the values of x, y, z, that will render the function

$$u = x^p y^q z^r$$

a maximum, the variables being subject to the condition

$$v = ax + by + cz - k = 0.$$

We find

$$\frac{du}{dx} = px^{p-1}y^q z^r = \frac{p}{x}u, \ \frac{du}{dy} = \frac{q}{y}u, \ \frac{du}{dz} = \frac{r}{z}u,$$

$$\frac{dv}{dx} = a, \ \frac{dv}{dy} = b, \ \frac{dv}{dz} = c:$$

therefore (Art. 128)

$$\frac{p}{ax} = \frac{q}{by} = \frac{r}{cz} = \frac{p+q+r}{k},$$

$$x = \frac{p}{a}\,\frac{k}{p+q+r}, \ y = \frac{q}{b}\,\frac{k}{p+q+r}, \ z = \frac{r}{c}\,\frac{k}{p+q+r}.$$

These values of x, y, z, make the function a maximum. For we find

$$\frac{d^2u}{dx^2} = \frac{p}{x}\frac{du}{dx} - \frac{p}{x^2}u, \ \frac{d^2u}{dy^2} = \frac{q}{y}\frac{du}{dy} - \frac{q}{y^2}u, \ \frac{d^2u}{dz^2} = \frac{r}{z}\frac{du}{dz} - \frac{r}{z^2}u;$$

and these, because $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$, $\frac{du}{dz} = 0$, for the values of x, y, z, become

$$\frac{d^2u}{dx^2} = -\frac{p}{x^2}u, \ \frac{d^2u}{dy^2} = -\frac{q}{y^2}u, \ \frac{d^2u}{dz^2} = -\frac{r}{z^2}u,$$

all of which are negative,—a necessary condition for a maximum; and, by getting the partial derivatives $\frac{d^2u}{dxdy}$, $\frac{d^2u}{dxdz}$, $\frac{d^2u}{dydz}$, we see that the other conditions (Art. 126) to insure this state of the function are also satisfied.

By making $a = 1$, $b = 1$, $c = 1$, the above becomes the solution of the problem for dividing the number k into three such parts, that the product of the p power of the first, the q power of the second, and the r power of the third, shall be a maximum.

8. Inscribe in a sphere the greatest parallelopipedon.

Ans. $\left\{\begin{array}{l}\text{If } a \text{ be the radius of the sphere, the parallelo-} \\ \text{pipedon is a cube having } \dfrac{2a}{\sqrt{3}} \text{ for its edge.}\end{array}\right.$

9. Determine a point within a triangle, from which, if lines be drawn to the vertices of the angles, the sum of their squares shall be a maximum.

Ans. The point is the intersection of the lines drawn from the vertices of the angles to the centres of the opposite sides.

A function $F(x, y, \ldots)$ of two or more variables may be of such form, that it admits of a maximum or minimum for values of the variables which make $\frac{dF}{dx}, \frac{dF}{dy}, \ldots$ indeterminate or infinite. There are no general rules applicable to such cases; but each one must be specially examined.

10. What values of x and y will make

$$u = ax^2 + (x^2 + by^2)^{\frac{1}{3}}$$

a minimum?

$$\frac{du}{dx} = 2ax + \frac{2}{3}\frac{x}{(x^2 + by^2)^{\frac{2}{3}}}, \quad \frac{du}{dy} = \frac{2}{3}\frac{y}{(x^2 + by^2)^{\frac{2}{3}}}.$$

For $x = 0$, $y = 0$, these differential co-efficients take the form $\frac{0}{0}$: but their true values are infinity; for, if we make $y = mx$, they become

$$\frac{du}{dx} = 2ax + \frac{2}{3}\frac{1}{x^{\frac{1}{3}}(1 + m^2b)^{\frac{2}{3}}}, \quad \frac{du}{dy} = \frac{2}{3}\frac{m}{x^{\frac{1}{3}}(1 + m^2b)^{\frac{2}{3}}}.$$

Hence for $x = 0$, and therefore for $y = 0$, at the same time, we have

$$\frac{du}{dx} = \infty, \quad \frac{du}{dy} = \infty.$$

For $x = 0$, $y = 0$, we have $u = 0$; and no real values of x and y can make u negative. Hence u is a minimum for $x = 0$, $y = 0$.

SECTION XIII.

CHANGE OF INDEPENDENT VARIABLES IN DIFFERENTIATION.

129. It is often required in investigations to change differential expressions, obtained under the supposition that certain variables were independent, into their equivalents when such variables are themselves functions of others.

Suppose that, having given $y = f(z)$, $x = \varphi(z)$, it is required to express the successive derivatives of y, taken as a function x, in terms of those of x and y taken with respect to z.

We have found (Art. 42)

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx},$$

and (Art. 41)

$$\frac{dz}{dx} = \frac{1}{\frac{dx}{dz}} : \quad \therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dz}}{\frac{dx}{dz}}.$$

Hence $$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{\frac{dy}{dz}}{\frac{dx}{dz}} = \frac{d}{dz}\frac{\frac{dy}{dz}}{\frac{dx}{dz}}\frac{dz}{dx} \quad \text{(Art. 42)}$$

$$= \frac{\frac{d^2y}{dz^2}\frac{dx}{dz} - \frac{d^2x}{dz^2}\frac{dy}{dz}}{\left(\frac{dx}{dz}\right)^2}\frac{dz}{dx}$$

$$= \frac{\frac{d^2y}{dz^2}\frac{dx}{dz} - \frac{d^2x}{dz^2}\frac{dy}{dz}}{\left(\frac{dx}{dz}\right)^3}, \text{ since } \frac{dz}{dx} = \frac{1}{\frac{dx}{dz}}.$$

So also

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{\frac{d^2y}{dz^2}\frac{dx}{dz} - \frac{d^2x}{dz^2}\frac{dy}{dz}}{\left(\frac{dx}{dz}\right)^3}$$

$$= \frac{d}{dz}\frac{\frac{d^2y}{dz^2}\frac{dx}{dz} - \frac{d^2x}{dz^2}\frac{dy}{dz}}{\left(\frac{dx}{dz}\right)^3}\frac{dz}{dx}$$

$$= \frac{\left(\frac{d^3y}{dz^3}\frac{dx}{dz} - \frac{d^3x}{dz^3}\frac{dy}{dz}\right)\left(\frac{dx}{dz}\right)^3 - 3\left(\frac{dx}{dz}\right)^2\frac{d^2x}{dz^2}\left(\frac{d^2y}{dz^2}\frac{dx}{dz} - \frac{d^2x}{dz^2}\frac{dy}{dz}\right)}{\left(\frac{dx}{dz}\right)^6}\frac{dz}{dx}$$

$$= \frac{\left(\frac{d^3y}{dz^3}\frac{dx}{dz} - \frac{d^3x}{dz^3}\frac{dy}{dz}\right)\frac{dx}{dz} - 3\frac{d^2x}{dz^2}\left(\frac{d^2y}{dz^2}\frac{dx}{dz} - \frac{d^2x}{dz^2}\frac{dy}{dz}\right)}{\left(\frac{dx}{dz}\right)^5}.$$

In the same manner, we may find $\frac{d^4y}{dx^4}$, $\frac{d^5y}{dx^5}\dots$ Substituting in these the values of $\frac{dx}{dz}$, $\frac{dy}{dz}$, $\frac{d^2x}{dz^2}\cdots$, found from

$$y = f(z),\ x = \varphi(z),$$

we have the values of the successive derivatives of y with respect to x, in terms of those of x and y with respect to z.

130. Having $y = f(x)$, to change the independent variable from x to y in the expressions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}\cdots$

Since $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \text{(Art. 41)},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{1}{\frac{dx}{dy}} = \frac{d}{dy}\frac{1}{\frac{dx}{dy}}\frac{dy}{dx} \quad \text{(Art. 42)},$$

$$= -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}\frac{dy}{dx} = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}.$$

Similarly,

$$\frac{d^3y}{dx^3} = -\frac{d}{dx}\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3} = -\frac{d}{dy}\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}\frac{dy}{dx}$$

$$= -\frac{\frac{d^3x}{dy^3}\left(\frac{dx}{dy}\right)^3 - 3\left(\frac{dx}{dy}\right)^2\left(\frac{d^2x}{dy^2}\right)^2}{\left(\frac{dx}{dy}\right)^7}$$

$$= -\frac{\frac{d^3x}{dy^3}\frac{dx}{dy} - 3\left(\frac{d^2x}{dy^2}\right)^2}{\left(\frac{dx}{dy}\right)^5}.$$

In like manner, we may find the expressions for $\frac{d^4y}{dx^4}, \frac{d^5y}{dx^5} \cdots$

These formulas may also be found from those in the preceding article, by making $z = y$; whence

$$\frac{dy}{dz} = 1, \quad \frac{d^2y}{dz^2} = 0, \quad \frac{d^3y}{dz^3} = 0 \ldots, \quad \frac{dx}{dz} = \frac{dx}{dy}, \quad \frac{d^2x}{dz^2} = \frac{d^2x}{dy^2} \cdots$$

By the introduction of these values in the formulas of Art. 129, they will be found to agree with those just established.

131. Having given $$y = f(x) \quad (1),$$
and also $$x = r\cos.\theta, \quad y = r\sin.\theta \quad (2),$$

it is evident that we may eliminate x and y from these equations, and get a direct relation between r and θ; and thus r becomes a function of θ.

It is required to express the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}\cdots$, derived from Eq. 1, in terms of $\frac{dr}{d\theta}$, $\frac{d^2r}{d\theta^2}\cdots$

By Arts. 41, 42, we have

$$\frac{dy}{dx}=\frac{dy}{d\theta}\frac{d\theta}{dx}=\frac{dy}{d\theta}\frac{1}{\frac{dx}{d\theta}}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$=\frac{\sin.\theta\frac{dr}{d\theta}+r\cos.\theta}{\cos.\theta\frac{dr}{d\theta}-r\sin.\theta}\text{ from Eqs. 2;}$$

also

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{\sin.\theta\frac{dr}{d\theta}+r\cos.\theta}{\cos.\theta\frac{dr}{d\theta}-r\sin.\theta}=\frac{d}{d\theta}\frac{\sin.\theta\frac{dr}{d\theta}+r\cos.\theta}{\cos.\theta\frac{dr}{d\theta}-r\sin.\theta}\frac{d\theta}{dx}.$$

Performing the indicated differentiation, we find for the numerator of the result

$$\left(\sin.\theta\frac{d^2r}{d\theta^2}+2\cos.\theta\frac{dr}{d\theta}-r\sin.\theta\right)\left(\cos.\theta\frac{dr}{d\theta}-r\sin.\theta\right)$$
$$-\left(\cos.\theta\frac{d^2r}{d\theta^2}-2\sin.\theta\frac{dr}{d\theta}-r\cos.\theta\right)\left(\sin.\theta\frac{dr}{d\theta}+r\cos.\theta\right),$$

which reduces to $\qquad r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2};$

and the denominator, remembering that $\frac{dx}{d\theta}=\cos.\theta\frac{dr}{d\theta}-r\sin.\theta$, is

$$\left(\cos.\theta\frac{dr}{d\theta}-r\sin.\theta\right)^3.$$

Hence
$$\frac{d^2y}{dx^2} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\cos.\theta\frac{dr}{d\theta} - r\sin.\theta\right)^3}.$$

These formulas are used in the applications of the differential calculus to geometry, where a change of reference is made from rectilinear to polar co-ordinates.

132. Suppose that we have the expressions for $\frac{du}{dx}, \frac{du}{dy}$, found from the equation $u = F(x, y)$; but that the variables x and y are connected with two other variables, r and θ, by the equations $x = F_1(r, \theta)$, $y = F_2(r, \theta)$: then we may conceive x and y to be eliminated from these three equations, and u to be a function of r and θ. Required the equivalents of the expressions for $\frac{du}{dx}, \frac{du}{dy}$, in terms of the derivatives of x, y, and u, with respect to r and θ.

By Art. 82, we have

$$\left.\begin{aligned} \frac{du}{dr} &= \frac{du}{dx}\frac{dx}{dr} + \frac{du}{dy}\frac{dy}{dr} \\ \frac{du}{d\theta} &= \frac{du}{dx}\frac{dx}{d\theta} + \frac{du}{dy}\frac{dy}{d\theta} \end{aligned}\right\} \quad (1);$$

and from these two equations the values of $\frac{du}{dx}, \frac{du}{dy}$, can be found.

When the equations expressing the relations between x, y, r, θ, are $r = F_1(x, y)$, $\theta = F_2(x, y)$, instead of those given above, then

$$\left.\begin{aligned} \frac{du}{dx} &= \frac{du}{dr}\frac{dr}{dx} + \frac{du}{d\theta}\frac{d\theta}{dx} \\ \frac{du}{dy} &= \frac{du}{dr}\frac{dr}{dy} + \frac{du}{d\theta}\frac{d\theta}{dy} \end{aligned}\right\} \quad (2).$$

If the variables x, y, r, θ, are connected by the unresolved equations $F_1(x, y, r, \theta) = 0$, $F_2(x, y, r, \theta) = 0$, we proceed thus:—

By Art. 82,

$$\left(\frac{dF_1}{d\theta}\right) + \frac{dF_1}{dx}\frac{dx}{d\theta} + \frac{dF_1}{dy}\frac{dy}{d\theta} = 0,$$

$$\left(\frac{dF_2}{d\theta}\right) + \frac{dF_2}{dx}\frac{dx}{d\theta} + \frac{dF_2}{dy}\frac{dy}{d\theta} = 0;$$

in which it must be remembered that $\left(\frac{dF_1}{d\theta}\right)$, $\left(\frac{dF_2}{d\theta}\right)$, are partial derivatives of F_1, F_2, with respect to θ.

Differentiating F_1, F_2, with respect to r, we get two similar equations involving $\frac{dx}{dr}$, $\frac{dy}{dr}$; and the four equations thus obtained will determine $\frac{dx}{d\theta}$, $\frac{dy}{d\theta}$, $\frac{dx}{dr}$, $\frac{dy}{dr}$, which must be substituted in formulas (1) or (2), Art. 131.

The following example will illustrate the manner of using the above formulas:—

Given $u = f(x, y)$, $x = r \cos.\theta$, $y = r \sin.\theta$, it is required to express $\frac{du}{dx}$, $\frac{du}{dy}$, in terms of r, θ, $\frac{du}{dr}$, $\frac{du}{d\theta}$.

We have

$$\frac{dx}{d\theta} = -r \sin.\theta, \quad \frac{dy}{d\theta} = r \cos.\theta,$$

$$\frac{dx}{dr} = \cos.\theta, \qquad \frac{dy}{dr} = \sin.\theta.$$

Hence, by formulas (1),

$$\frac{du}{dr} = \cos.\theta \frac{du}{dx} + \sin.\theta \frac{du}{dy},$$

$$\frac{du}{d\theta} = -r \sin.\theta \frac{du}{dx} + r \cos.\theta \frac{du}{dy}:$$

whence
$$\left.\begin{aligned}\frac{du}{dx} &= \cos.\theta\frac{du}{dr} - \frac{1}{r}\sin.\theta\frac{du}{d\theta}\\ \frac{du}{dy} &= \sin.\theta\frac{du}{dr} + \frac{1}{r}\cos.\theta\frac{du}{d\theta}\end{aligned}\right\} \quad (a).$$

To make formulas (2) applicable to this example, we first deduce, from the equations $x = r\cos.\theta$, $y = r\sin.\theta$, the values of r and θ in terms of x and y. We find

$$r = \sqrt{x^2 + y^2},\ \theta = \tan.^{-1}\frac{y}{x};$$

whence $\dfrac{dr}{dx} = \dfrac{x}{r},\ \dfrac{dr}{dy} = \dfrac{y}{r},\ \dfrac{d\theta}{dx} = -\dfrac{y}{r^2},\ \dfrac{d\theta}{dy} = \dfrac{x}{r^2}:$

and, by means of these, Eqs. 2 become

$$\left.\begin{aligned}\frac{du}{dx} &= \frac{x}{r}\frac{du}{dr} - \frac{y}{r^2}\frac{du}{d\theta}\\ \frac{du}{dy} &= \frac{y}{r}\frac{du}{dr} + \frac{x}{r^2}\frac{du}{d\theta}\end{aligned}\right\} \quad (b).$$

The relations $x = r\cos.\theta$, $y = r\sin.\theta$, give $\cos.\theta = \dfrac{x}{r}$, $\sin.\theta = \dfrac{y}{r}$, by means of which we can pass from formulas (b) to (a), or the opposite.

133. Attention is here called to the necessity of attaching their precise signification to the symbols

$$\frac{dr}{dx},\ \frac{dr}{dy},\ \frac{dx}{dr},\ \frac{dy}{dr},\ \frac{d\theta}{dx},\ \frac{d\theta}{dy},\ \frac{dx}{d\theta},\ \frac{dy}{d\theta},$$

which occur in formulas (1) and (2), Art. 131.

It must be borne in mind that these denote partial differential co-efficients, and that those referring to the same variables, such as $\dfrac{dr}{dx}$, $\dfrac{dx}{dr}$, have not to each other the relation of $\dfrac{dy}{dx}$ to $\dfrac{dx}{dy}$, which are derived from the equation $f(x, y) = 0$. With reference to these last, we know that one is the recipro-

cal of the other, or that their product is 1; but this is not true for $\frac{dr}{dx} \times \frac{dx}{dr}$. The consideration of the meaning of the term "differential co-efficient," and the difference between the equations connecting the variables in the two cases, will remove all difficulty. In getting formulas (1), x and y were given as explicit functions of the independent variables r and θ; and a change in either r or θ will produce changes in both x and y. Hence, in the operation of finding $\frac{dx}{dr}$, r, x, and y vary, while θ remains constant. In formulas (2), r and θ were given as explicit functions of x and y; and a change in the value of either x or y will produce changes in both r and θ; and hence the increment attributed to x in getting $\frac{dr}{dx}$ causes r and θ also to vary, while y remains constant. That is, in formulas (1), $\frac{dx}{dr}$ supposes r, x, and y to vary together, θ being constant; while, in formulas (2), $\frac{dr}{dx}$ supposes x, r, and θ to vary, while y remains constant. Thus it appears that these two partial derivatives are obtained on different suppositions in respect to the variables which receive increments, and those which remain constant.

In the example just given for formulas (1), we have $\frac{dx}{dr} = \cos.\theta$; and, for formulas (2), $\frac{dr}{dx} = \cos.\theta$; and the product $\frac{dx}{dr}\frac{dr}{dx} = \cos.^2\theta$.

134. Having $u = F(x, y, z)$, and three equations expressing the relations between x, y, z, and three other variables r, θ, ψ, it is required to find the values of $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, in terms of the different co-efficients of u with respect to r, θ, ψ.

By Art. 82,

$$\left.\begin{aligned}\frac{du}{dx}&=\frac{du}{d\theta}\frac{d\theta}{dx}+\frac{du}{d\psi}\frac{d\psi}{dx}+\frac{du}{dr}\frac{dr}{dx}\\ \frac{du}{dy}&=\frac{du}{d\theta}\frac{d\theta}{dy}+\frac{du}{d\psi}\frac{d\psi}{dy}+\frac{du}{dr}\frac{dr}{dy}\\ \frac{du}{dz}&=\frac{du}{d\theta}\frac{d\theta}{dz}+\frac{du}{d\psi}\frac{d\psi}{dz}+\frac{du}{dr}\frac{dr}{dz}\end{aligned}\right\}\quad(1).$$

The three equations connecting x, y, z, r, θ, ψ, will enable us to determine $\frac{d\theta}{dx}, \frac{d\theta}{dy}, \frac{d\theta}{dz}, \frac{dr}{dx}, \ldots \frac{d\psi}{dx} \ldots$; and Eqs. 1, when these values are substituted in them, give us the expressions sought.

By solving Eqs. 1, we can also find the values of $\frac{du}{d\theta}, \frac{du}{dr}, \frac{du}{d\psi}$, expressed in terms of $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$; or we may find these values from the equations

$$\left.\begin{aligned}\frac{du}{d\theta}&=\frac{du}{dx}\frac{dx}{d\theta}+\frac{du}{dy}\frac{dy}{d\theta}+\frac{du}{dz}\frac{dz}{d\theta}\\ \frac{du}{d\psi}&=\frac{du}{dx}\frac{dx}{d\psi}+\frac{du}{dy}\frac{dy}{d\psi}+\frac{du}{dz}\frac{dz}{d\psi}\\ \frac{du}{dr}&=\frac{du}{dx}\frac{dx}{dr}+\frac{du}{dy}\frac{dy}{dr}+\frac{du}{dz}\frac{dz}{dr}\end{aligned}\right\}\quad(2).$$

135. Let the relations between the variables x, y, z, θ, ψ, r, be

$$x = r \sin.\theta \cos.\psi, \quad y = r \sin.\theta \sin.\psi, \quad z = r \cos.\theta \quad (1').$$

From these we find

$$\frac{dx}{d\theta} = r \cos.\theta \cos.\psi, \quad \frac{dy}{d\theta} = r \cos.\theta \sin.\psi, \quad \frac{dz}{d\theta} = -r \sin.\theta,$$

$$\frac{dx}{d\psi} = -r \sin.\theta \sin.\psi, \quad \frac{dy}{d\psi} = r \sin.\theta \cos.\psi, \quad \frac{dz}{d\psi} = 0,$$

$$\frac{dx}{dr} = \sin.\theta \cos.\psi, \quad \frac{dy}{dr} = \sin.\theta \sin.\psi, \quad \frac{dz}{dr} = \cos.\theta;$$

and formulas (2), Art. 134, by the substitution of these values, become

$$\left.\begin{aligned}
\frac{du}{d\theta} &= r\cos.\theta\cos.\psi\frac{du}{dx} + r\cos.\theta\sin.\psi\frac{du}{dy} - r\sin.\theta\frac{du}{dz} \\
\frac{du}{d\psi} &= -r\sin.\theta\sin.\psi\frac{du}{dx} + r\sin.\theta\cos.\psi\frac{du}{dy} \\
\frac{du}{dr} &= \sin.\theta\cos.\psi\frac{du}{dx} + \sin.\theta\sin.\psi\frac{du}{dy} + \cos.\theta\frac{du}{dz}
\end{aligned}\right\} \quad (a).$$

From Eqs. a may be found the required values of $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, in terms of $\frac{du}{d\theta}, \frac{du}{d\psi}, \frac{du}{dr}$.

Again: squaring Eqs. 1′, adding results, and taking square root, we have $r = \sqrt{(x^2 + y^2 + z^2)}$. Adding the squares of first and second of these equations, we find $r^2 \sin.^2 \theta = x^2 + y^2$; whence $r\sin.\theta = \sqrt{(x^2 + y^2)}$, $\sin.\theta = \frac{1}{r}\sqrt{(x^2 + y^2)}$: and from this, and the last of Eqs. 1′, we find

$$\tan.\theta = \frac{\sqrt{(x^2 + y^2)}}{z},\quad \theta = \tan.^{-1}\frac{\sqrt{(x^2 + y^2)}}{z}.$$

Dividing the second of Eqs. 1′ by the first, we have

$$\tan.\psi = \frac{y}{x},\quad \psi = \tan.^{-1}\frac{y}{x}.$$

Hence we have

$$r = \sqrt{(x^2 + y^2 + z^2)},\quad \theta = \tan.^{-1}\frac{\sqrt{x^2 + y^2}}{z},\quad \psi = \tan.^{-1}\frac{y}{x} \quad (2').$$

From those by differentiation, we have

$$\frac{dr}{dx} = \frac{x}{r} = \sin.\theta\cos.\psi,\quad \frac{dr}{dy} = \frac{y}{r} = \sin.\theta\sin.\psi,\quad \frac{dr}{dz} = \frac{z}{r} = \cos.\theta,$$

$$\frac{d\theta}{dx} = \frac{z}{x^2 + y^2 + z^2} \times \frac{x}{\sqrt{x^2 + y^2}} = \frac{\cos.\theta\cos.\psi}{r},$$

$$\frac{d\theta}{dy} = \frac{z}{x^2 + y^2 + z^2} \times \frac{y}{\sqrt{x^2 + y^2}} = \frac{\cos.\theta\sin.\psi}{r},$$

$$\frac{d\theta}{dz}=-\frac{\sqrt{(x^2+y^2)}}{x^2+y^2+z^2}=-\frac{\sin.\theta}{r},$$

$$\frac{d\psi}{dx}=-\frac{y}{x^2+y^2}=-\frac{\sin.\psi}{r\sin.\theta},\ \frac{d\psi}{dy}=\frac{x}{x^2+y^2}=\frac{\cos.\psi}{r\sin.\theta},\ \frac{d\psi}{dz}=0.$$

By the substitution of these values in formulas (1), Art. 134, we have

$$\left.\begin{aligned}\frac{du}{dx}&=\frac{\cos.\theta\cos.\psi}{r}\frac{du}{d\theta}-\frac{\sin.\psi}{r\sin.\theta}\frac{du}{d\psi}+\sin.\theta\cos.\psi\frac{du}{dr}\\ \frac{du}{dy}&=\frac{\cos.\theta\sin.\psi}{r}\frac{du}{d\theta}+\frac{\cos.\psi}{r\sin.\theta}\frac{du}{d\psi}+\sin.\theta\sin.\psi\frac{du}{dr}\\ \frac{du}{dz}&=-\frac{\sin.\theta}{r}\frac{du}{d\theta}+\cos.\theta\frac{du}{dr}\end{aligned}\right\}\quad(b).$$

The values $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, given by formulas ($a$), will be found to agree with those given directly by formulas (b).

EXAMPLES.

1. Transform

$$x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3-\frac{dy}{dx}=0 \quad (1)$$

into its equivalent when neither x nor y is independent, but both are functions of a third variable z.

Substitute for $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ their values given in Art. 129, and we have

$$x\frac{\frac{d^2y}{dz^2}\frac{dx}{dz}-\frac{d^2x}{dz^2}\frac{dy}{dz}}{\left(\frac{dx}{dz}\right)^3}+\left\{\frac{\frac{dy}{dz}}{\frac{dx}{dz}}\right\}^3-\frac{\frac{dy}{dz}}{\frac{dx}{dz}}=0;$$

and, multiplying through by $\left(\frac{dx}{dz}\right)^3$,

$$x\frac{d^2y}{dz^2}\frac{dx}{dz}-x\frac{d^2x}{dz^2}\frac{dy}{dz}+\left(\frac{dy}{dz}\right)^3-\frac{dy}{dz}\left(\frac{dx}{dz}\right)^2=0 \quad (2).$$

If we make $x = z$, this reduces to the given equation. Making $y = z$, (2) becomes

$$x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 - 1 = 0 \quad (3).$$

Equation (3) is the equivalent of (1) when the independent variable is changed from x to y.

2. Change

$$\left\{\left(\frac{dy}{dx}\right)^2 + 1\right\}^{\frac{3}{2}} + a\frac{d^2y}{dx^2} = 0$$

into its equivalent when both x and y are functions of a third variable z.

$$\text{Ans. } \left\{\left(\frac{dy}{dz}\right)^2 + \left(\frac{dx}{dz}\right)^2\right\}^{\frac{3}{2}} + a\left(\frac{d^2y}{dz^2}\frac{dx}{dz} - \frac{d^2x}{dz^2}\frac{dy}{dz}\right) = 0.$$

If $y = z$, the above becomes

$$\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}} - a\frac{d^2x}{dy^2} = 0,$$

in which y is the independent variable.

3. Eliminate x between

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0, \text{ and } x^2 = 4\theta \quad (1),$$

and find what the differential equation is when θ is the independent variable, and also when y is the independent variable.

First suppose both x and y to be functions of a third variable, z; then the differential equation becomes (Art. 129)

$$\frac{d^2y}{dz^2}\frac{dx}{dz} - \frac{d^2x}{dz^2}\frac{dy}{dz} + \frac{1}{x}\frac{dy}{dz}\left(\frac{dx}{dz}\right)^2 + y\left(\frac{dx}{dz}\right)^3 = 0 \quad (2).$$

From $x^2 = 4\theta$, we have $x = 2\theta^{\frac{1}{2}}$, $\frac{dx}{dz} = \frac{dx}{d\theta}\frac{d\theta}{dz}$: but $\frac{dx}{d\theta} = \theta^{-\frac{1}{2}}$;

$$\therefore \frac{dx}{dz} = \theta^{-\frac{1}{2}}\frac{d\theta}{dz}, \quad \frac{d^2x}{dz^2} = \frac{d}{dz}\theta^{-\frac{1}{2}}\frac{d\theta}{dz} = \theta^{-\frac{1}{2}}\frac{d^2\theta}{dz^2} - \frac{1}{2}\theta^{-\frac{3}{2}}\left(\frac{d\theta}{dz}\right)^2.$$

In (2) replacing $x, \frac{dx}{dz}, \frac{d^2x}{dz^2}$, by their values, we have

$$\theta^{-\frac{1}{2}}\frac{d\theta}{dz}\frac{d^2y}{dz^2} - \theta^{-\frac{1}{2}}\frac{d^2\theta}{dz^2}\frac{dy}{dz} + \theta^{-\frac{3}{2}}\left(\frac{d\theta}{dz}\right)^2\frac{dy}{dz} + y\theta^{-\frac{3}{2}}\left(\frac{d\theta}{dz}\right)^3 = 0 \quad (3),$$

which does not contain x. Making $y = z$, (3) becomes

$$\theta\frac{d^2\theta}{dy^2} - \left(\frac{d\theta}{dy}\right)^2 - y\left(\frac{d\theta}{dy}\right)^3 = 0 \quad (4);$$

and if, instead, $\theta = z$, we have

$$\theta\frac{d^2y}{d\theta^2} + \frac{dy}{d\theta} + y = 0 \quad (5).$$

4. Given the relation $x = e^s$, to change the independent variable, in the differential expression $x^n\frac{d^ny}{dx^n}$, from x to s.

By Art. 42, we have

$$\frac{d}{ds}\left(x^n\frac{d^ny}{dx^n}\right) = \frac{d}{dx}\left(x^n\frac{d^ny}{dx^n}\right)\frac{dx}{ds} = \left(nx^{n-1}\frac{d^ny}{dx^n} + x^n\frac{d^{n+1}y}{dx^{n+1}}\right)x$$

$$= nx^n\frac{d^ny}{dx^n} + x^{n+1}\frac{d^{n+1}y}{dx^{n+1}}:$$

$$\therefore \frac{d}{ds}\left(x^n\frac{d^ny}{dx^n}\right) - nx^n\frac{d^ny}{dx^n} = x^{n+1}\frac{d^{n+1}y}{dx^{n+1}};$$

or, writing the first member in an abbreviated form,

$$\left(\frac{d}{ds} - n\right)x^n\frac{d^ny}{dx^n} = x^{n+1}\frac{d^{n+1}y}{dx^{n+1}} \quad (1).$$

Making $n = 1$, this gives

$$\left(\frac{d}{ds} - 1\right)x\frac{dy}{dx} = x^2\frac{d^2y}{dx^2} \quad (2).$$

From $x = e^s$, we get $\frac{dx}{ds} = e^s = x$; also we have

$$\frac{dy}{ds} = \frac{dy}{dx}\frac{dx}{ds} = x\frac{dy}{dx}:$$

hence (2) becomes

$$x^2\frac{d^2y}{dx^2} = \left(\frac{d}{ds} - 1\right)\frac{dy}{ds} \quad (3).$$

When $n = 2$ in formula (1), then

$$\left(\frac{d}{ds} - 2\right)x^2\frac{d^2y}{dx^2} = x^3\frac{d^3y}{dx^3};$$

and, putting in this the value of $x^2\dfrac{d^2y}{dx^2}$ from (3),

$$x^3\frac{d^3y}{dx^3} = \left(\frac{d}{ds} - 1\right)\left(\frac{d}{ds} - 2\right)\frac{dy}{ds}.$$

The law governing the construction of these equations is obvious; and we may write, generally,

$$x^n\frac{d^ny}{dx^n} = \left(\frac{d}{ds} - 1\right)\left(\frac{d}{ds} - 2\right)\dots\left(\frac{d}{ds} - (n-2)\right)\left(\frac{d}{ds} - (n-1)\right)\frac{dy}{ds} \quad (4).$$

The meaning of the operations denoted in the second member of formula (4) is, that if the expressions $\dfrac{d}{ds} - 1$, $\dfrac{d}{ds} - 2\dots$, be combined by the rules for multiplication, the result will represent, in terms of indicated differentiations on $\dfrac{dy}{ds}$, the value of $x^n\dfrac{d^ny}{dx^n}$.

5. If we have $\rho = -\dfrac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$, and the relations $x = r\cos.\theta$, $y = r\sin.\theta$, find the equivalent for ρ when a change of independent variable is made from x to θ, and also from x to r.

When θ is the independent variable,

$$\rho = \frac{\left\{\left(\dfrac{dr}{d\theta}\right)^2 + r^2\right\}^{\frac{3}{2}}}{r\dfrac{d^2r}{d\theta^2} - 2\left(\dfrac{dr}{d\theta}\right)^2 - r^2};$$

and, when r is independent,

$$\rho = -\frac{\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{\frac{3}{2}}}{r\frac{d^2\theta}{dr^2} + r\frac{d\theta}{dr} + 2\frac{d\theta}{dr}}.$$

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SECTION XIV.

ELIMINATION OF CONSTANTS AND ARBITRARY FUNCTIONS BY DIFFERENTIATION.

136. WHEN an equation is given in the form

$$F(x, y) = c \quad (1),$$

the constant c will disappear on the first differentiation, and the successive differential equations derived from (1) will be identical with those derived from

$$F(x, y) = 0 \quad (2).$$

Though an equation may not be given under the form of (1), it often happens that one or more of its constants may be made to disappear by successive differentiation alone.

Let $$(y - b)^2 + (x - a)^2 - r^2 = 0 \quad (3),$$

and differentiate this equation twice, taking x as the independent variable. We find

$$(y - b)\frac{dy}{dx} + x - a = 0 \qquad (4),$$

$$(y - b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0 \quad (5);$$

and thus the two constants a and r of (3) have vanished in the two differentiations which lead to (5). A third differentiation gives

$$(y - b)\frac{d^3y}{dx^3} + 3\frac{dy}{dx}\frac{d^2y}{dx^2} = 0 \quad (6).$$

From Eqs. 3, 4, 5, and 6, we get

$$\frac{dy}{dx}=\frac{x-a}{b-y},\ \frac{d^2y}{dx^2}=-\frac{(x-a)^2+(b-y)^2}{(b-y)^3}=\frac{r^2}{(b-y)^3},$$

$$\frac{d^3y}{dx^3}=-\frac{3\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}}{y-b}=\frac{3r^2(x-a)}{(b-y)^5};$$

and, by eliminating $y-b$ between (5) and (6), we get

$$\left\{\left(\frac{dy}{dx}\right)^2+1\right\}\frac{d^3y}{dx^3}-3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2=0 \quad (7).$$

Between (3) and (4), we may eliminate any one of three constants a, b, r; and, by taking these constants in succession, we should have for our results three differential equations of the first order, each containing two of the constants. By a proper combination of (3), (4), and (5), we can arrive at two differential equations of the second order, each containing but one of the constants of the primitive equation; and between (3), (4), (5), and (6), we can eliminate all three of the constants, obtaining for the result a single differential equation of the third order.

It thus appears, that, by differentiation and elimination, Eq. 3 will give rise, 1st, To three differential equations of the first order, each involving two of the constants a, b, r; 2d, To two differential equations of the second order, each involving but one of these constants; 3d, To one differential equation of the third order, from which all of the constants have vanished.

By means of Eqs. 3, 4, 5, the values of a, b, r, may be expressed in terms of x, y, and the derivatives of y of the first and second orders. Denoting these derivatives by y', y'', we find

$$b=y+\frac{y'^2+1}{y''},\ a=x-\frac{y'(y'^2+1)}{y''},\ r=\pm\frac{(y'^2+1)^{\frac{3}{2}}}{y''}.$$

137. In general, if we have an equation between x and y, and n arbitrary constants, and we differentiate this equation m times successively, we shall have, with the primitive equation, $m+1$ equations, between which we can eliminate m constants. This will lead to a differential equation of the m^{th} order, in which there will be but $n-m$ of the constants; and, as the constants eliminated may be selected at pleasure, it is evident that as many equations of the order m may be formed, each containing $n-m$ constants, as we can form combinations of n things taken m in a set, which is expressed by

$$\frac{n(n-1)(n-2)\ldots(n-m+1)}{1.2.3\ldots m}.$$

When the original equation is differentiated n times, we should have altogether $n+1$ equations, between which the n constants can be eliminated; and, as the resulting equation would involve the n^{th} differential co-efficient of y taken with respect to x, it is said to be of the n^{th} order. The order of the highest differential co-efficient entering any of the equations at which we arrive, by the steps above indicated, determines the order of the differential equation.

It is worthy of remark, that if any one of the differential equations of the m^{th} order, obtained by eliminating between the first m derived equations, and the primitive equation, m of the constants entering the latter, be differentiated $n-m$ times in succession, then this equation of the m^{th} order, and its $n-m$ derived equations, would enable us to eliminate the remaining constants; and the final equation at which we should arrive would be the same as that obtained by effecting the elimination between the primitive equation and its n successive derived equations.

To illustrate, take the equation $a = x - \frac{y'^2 + 1}{y''}$, and differentiate with respect to x. We should find, after reduction,

$$y'''(y'^2 + 1) - 3y'y''^2 = 0,$$

which agrees with Eq. 7.

The theory of the elimination of constants by differentiation is sufficiently simple, and needs but little explanation. It is referred to here for the reason that a knowledge of the formation of differential equations assists in understanding the more difficult and highly important operation of passing back from such equations to those from which it may be presumed that they have been derived.

138. Functions known and arbitrary may also be eliminated by differentiation.

Let $y = a \sin. x$; then $\frac{dy}{dx} = a \cos. x = \sqrt{a^2 - y^2}$:

$$\therefore \qquad y^2 + \left(\frac{dy}{dx}\right)^2 - a^2 = 0;$$

an equation which no longer contains the known function $\sin. x$.

Again: suppose $z = \varphi\left(\frac{x}{y}\right)$, in which x and y are independent, and φ denotes a function of the ratio of these variables, the form of which is not given, and is therefore called an *arbitrary function*.

Make $t = \frac{x}{y}$; then $z = \varphi(t)$, $\frac{dz}{dx} = \varphi'(t)\frac{dt}{dx} = \frac{1}{y}\varphi'(t)$,

$$\frac{dz}{dy} = \varphi'(t)\frac{dt}{dy} = -\frac{x}{y^2}\varphi'(t):$$

$$\therefore \qquad x\frac{dz}{dx} + y\frac{dz}{dy} = 0.$$

This last equation is true, whatever may be the form of the function of $\frac{x}{y}$ denoted by φ; it may be $z = l\left(\frac{x}{y}\right)$, $z = \sin.\frac{x}{y}$, or $z = e^{\frac{x}{y}}$: and for each of these cases the differential equation subsists.

Take the more general case, $u = \varphi(v)$, in which u and v are known functions of the independent variables x and y, and of the dependent variable z, and $\varphi(v)$ an arbitrary function of v. Differentiating $u = \varphi(v)$ first with respect to x and z, and then with respect to y and z, and, for brevity, making $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$, we shall have

$$\frac{du}{dx} + p\frac{du}{dz} = \varphi'(v)\left(\frac{dv}{dx} + p\frac{dv}{dz}\right),$$

$$\frac{du}{dy} + q\frac{du}{dz} = \varphi'(v)\left(\frac{dv}{dy} + q\frac{dv}{dz}\right).$$

Dividing these equations member by member, we have

$$\frac{\frac{du}{dx} + p\frac{du}{dz}}{\frac{du}{dy} + q\frac{du}{dz}} = \frac{\frac{dv}{dx} + p\frac{dv}{dz}}{\frac{dv}{dy} + q\frac{dv}{dz}}.$$

Clearing of fractions, and making

$$P = \frac{du}{dy}\frac{dv}{dz} - \frac{du}{dz}\frac{dv}{dy},\ Q = \frac{du}{dz}\frac{dv}{dx} - \frac{du}{dx}\frac{dv}{dz},\ R = \frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx},$$

we find that the partial differential co-efficients of the first order are connected by the equation

$$Pp + Qq = R:$$

and this equation is in no wise dependent upon the form of the function characterized by φ; in other words, this function has been eliminated.

139. Suppose c and c_1 to be two known functions of x, y, z,

expressed by $c = f(x, y, z)$, $c_1 = f_1(x, y, z)$; and that, in the equation

$$F\Big(x, y, z, \varphi(c), \varphi_1(c_1)\Big) = 0 \quad (1),$$

φ, φ_1, denote arbitrary functions. Let us see if it be possible to pass from (1) to a differential equation which shall not contain $\varphi(c)$, $\varphi_1(c_1)$, or their derivatives.

The equations

$$\frac{dF}{dx} = 0, \ \frac{dF}{dy} = 0 \quad (2),$$

$$\frac{d^2F}{dx^2} = 0, \ \frac{d^2F}{dxdy} = 0, \ \frac{d^2F}{dy^2} = 0 \quad (3),$$

that we get by differentiating (1), will contain the unknown functions $\varphi'(c)$, $\varphi_1'(c_1)$, $\varphi''(c)$, $\varphi_1''(c_1)$, which, with $\varphi(c)$, $\varphi_1(c_1)$, make six quantities to be eliminated between Eq. 1 and the five equations of groups (2) and (3), which are generally insufficient. Passing to the equations

$$\frac{d^3F}{dx^3} = 0, \ \frac{d^3F}{dx^2dy} = 0, \ \frac{d^3F}{dxdy^2} = 0, \ \frac{d^3F}{dy^3} = 0 \quad (4),$$

we introduce two additional arbitrary functions $\varphi'''(c)$, $\varphi_1'''(c_1)$, and only these two. We shall now have ten equations, viz. Eq. 1, and those of groups (2), (3), (4), and but eight arbitrary functions to eliminate: hence the elimination can be effected, and we may have two resulting differential equations of the third order.

We have said, that, in the case supposed above, it is generally impossible to effect the desired eliminations without passing to Eqs. 4. It may happen, however, that the forms of the functions $f(x, y, z)$, $f_1(x, y, z)$, are such that Eqs. 1, 2, 3, will prove sufficient.

Suppose $z = \varphi(x + ay) + \varphi_1(x - ay)$; then

$$\frac{dz}{dx} = \varphi'(x + ay) + \varphi_1'(x - ay),$$

$$\frac{dz}{dy} = a\varphi'(x + ay) - a\varphi'(x - ay),$$

$$\frac{d^2z}{dx^2} = \varphi''(x + ay) + \varphi_1''(x - ay),$$

$$\frac{d^2z}{dy^2} = a^2\varphi''(x + ay) + a^2\varphi_1''(x - ay).$$

From the last two of these equations, we find

$$\frac{d^2z}{dy^2} = a^2\frac{d^2z}{dx^2}.$$

140. Suppose that we have two functions,

$$F\Big(x, y, z, c, \varphi(c), \varphi_1(c)\ldots\Big) = 0,\ \ F_1\Big(x, y, z, c, \varphi(c), \varphi_1(c)\ldots\Big) = 0,$$

in which c is an implicit function of x, y, z, and $\varphi(c), \varphi_1(c)\ldots$, are arbitrary functions of c. It is proposed by successive differentiations to eliminate c and the arbitrary functions. To accomplish this, z and c must be considered as functions of the independent variables x, y; then, having differentiated the given equations a number of times successively with respect to x, and also with respect to y, we must eliminate the quantities

$$c,\ \frac{dc}{dx},\ \frac{dc}{dy},\ \frac{d^2c}{dx^2},\ \frac{d^2c}{dxdy},\ \frac{d^2c}{dy^2}\ldots \quad (1),$$

$$\varphi(c),\ \varphi'(c),\ \varphi''(c)\ldots,\ \varphi_1(c),\ \varphi_1'(c),\ \varphi_1''(c)\ldots \quad (2),$$

between the given and the differential equations.

Let m denote the number of arbitrary functions

$$\varphi(c),\ \varphi_1(c),\ \varphi_2(c)\ldots,$$

and n any positive integer; then, if we stop with partial

derivatives of c and of $\varphi(c)$, $\varphi_1(c)$, of the n^{th} order, the number of terms in series (1) will be expressed by

$$1+2+3+4+\cdots n+1=\frac{(n+1)(n+2)}{2}.$$

The number of the arbitrary functions

$$\varphi(c),\ \varphi'(c)\ldots,\ \varphi^{(n)}(c),\ \varphi_1(c),\ \varphi_1'(c)\ldots,\ \varphi_1^{(n)}(c)\ldots,$$

will be equal to $m(n+1)$. Again: since each of the given equations will give rise to two derived equations of the first order, three of the second, four of the third, and so on, the number of given and derived equations together will be equal to $(n+1)(n+2)$. Hence to be able, in the general case, to eliminate c and its arbitrary functions, and their derivatives up to the n^{th} order, we must have

$$(n+1)(n+2) > \frac{(n+1)(n+2)}{2}+(n+1)m, \text{ or } \frac{n}{2}+1>m.$$

This condition will be satisfied if $n=2m-1$, which will give $2m(2m+1)$ equations between which to eliminate $4m^2+m$ quantities. The number of equations exceeds the number of quantities to be eliminated by m: hence there will be, in general, m resulting differential equations.

When the proposed equations contain but one arbitrary function, $\varphi(c)$, of c, they become

$$F\big(x, y, z, c, \varphi(c)\big)=0,\ F_1\big(x, y, z, c, \varphi(c)\big)=0,$$

each of which gives two partial derived equations of the first order; and we shall thus have, including the given equations, six equations between the quantities

$$x,\ y,\ z,\ p=\frac{dz}{dx},\ q=\frac{dz}{dy},\ c,\ \frac{dc}{dx},\ \frac{dc}{dy},\ \varphi(c),\ \varphi'(c),$$

the elimination of the last five of which will lead to a single

partial differential equation of the first order between the variables x, y, z, of which x and y are independent.

If there are but two arbitrary functions $\varphi(c)$, $\varphi_1(c)$ of c, we should find that the given equations

$$F\Big(x, y, z, c, \varphi(c), \varphi_1(c)\Big) = 0, \quad F_1\Big(x, y, z, c, \varphi(c), \varphi_1(c)\Big) = 0,$$

with their partial derived equations of the first order, making in all twelve equations, would involve twelve quantities to be eliminated; viz.,

$$c, \frac{dc}{dx}, \frac{dc}{dy}, \frac{d^2c}{dx^2}, \frac{d^2c}{dxdy}, \frac{d^2c}{dy^2},$$

$$\varphi(c), \varphi'(c), \varphi''(c), \varphi_1(c), \varphi_1'(c), \varphi_1''(c):$$

hence the elimination cannot be effected, except in special cases. Passing to the partial derived equations of the third order, we should then have in all twenty equations, with eighteen quantities to be eliminated; viz., the twelve above given, and

$$\frac{d^3c}{dx^3}, \frac{d^3c}{dx^2dy}, \frac{d^3c}{dxdy^2}, \frac{d^3c}{dy^3}, \varphi'''(c), \varphi_1'''(c),$$

additional: and we may therefore have for our results two partial differential equations of the third order between x, y, z; the latter being the dependent variable.

In certain cases, it is unnecessary to make as many differentiations as have been indicated to enable us to effect the desired eliminations. Suppose, for example, that the given equations contain but three arbitrary functions, $\varphi(c)$, $\varphi_1(c)$, $\varphi_2(c)$: in this case, $m = 3$, $2m - 1 = 5$; and, to effect the eliminations, it would be generally necessary to form the derived equations of the fifth order, and we should have for our results three partial differential equations of the fifth order between x, y, z. But if the arbitrary functions are so related

that $\varphi_1(c) = \varphi'(c)$, $\varphi_2(c) = \varphi''(c)$, the proposed equations become

$$F\left\{x, y, z, c, \varphi(c), \varphi'(c), \varphi''(c)\right\} = 0,$$
$$F_1\left\{x, y, z, c, \varphi(c), \varphi'(c), \varphi''(c)\right\} = 0;$$

and these, with their derived equations of the first and second orders, make twelve equations, involving the eleven quantities

$$c, \frac{dc}{dx}, \frac{dc}{dy}, \frac{d^2c}{dx^2}, \frac{d^2c}{dxdy}, \frac{d^2c}{dy^2},$$
$$\varphi(c), \varphi'(c), \varphi''(c), \varphi'''(c), \varphi''''(c);$$

and the elimination will lead to a single partial differential equation of the second order.

If the value c be found, as it may be, theoretically at least, from one, say the second, of the equations

$$F\left\{x, y, z, c, \varphi(c), \varphi_1(c)\right\} = 0, \ F_1\left\{x, y, z, c, \varphi(c), \varphi_1(c)\right\} = 0,$$

and this value be substituted in the first, we should have for our result an equation of the form

$$F\left\{x, y, z, \psi(x, y, z), \psi_1(x, y, z)\right\} = 0,$$

which is evidently equivalent to the two proposed equations. By Art. 139, we shall generally be unable to eliminate the two arbitrary functions ψ, ψ_1, with this equivalent equation and its derived equations of the first and second orders; but it would be necessary to pass to the third derived equations to effect the elimination.

EXAMPLES.

1. Eliminate the constant a from the equation

$$\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y).$$

Ans. $\dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}} - \dfrac{dy}{dx} = 0.$

2. Eliminate c from the equation

$$x^2 + y^2 = cx.$$

Ans. $y^2 - 2xy\dfrac{dy}{dx} - x^2 = 0.$

3. Eliminate the functions e^x and $\cos. x$ from

$$y - e^x \cos. x = 0.$$

Ans. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0.$

4. From $y = a\sin. x + b\cos. x$ eliminate the functions $\sin. x$, $\cos. x$.

Ans. $\dfrac{d^2y}{dx^2} + y = 0.$

5. If $y = ce^{\sin.^{-1}x}$, prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0.$$

6. If $y = be^{ax}\cos. (nx + c)$, show that

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + n^2)\, y = 0.$$

7. From the equation $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ eliminate the exponential functions.

Ans. $y^2 + \dfrac{dy}{dx} - 1 = 0.$

8. From $z = \varphi(e^x \sin. y)$ eliminate the arbitrary function characterized by φ.

Ans. $\sin. y\dfrac{dz}{dy} - \cos. y\dfrac{dz}{dx} = 0.$

9. From $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1 = 0$ eliminate the constants a, b, c.

1st Ans. $xz\dfrac{d^2z}{dx^2} + x\left(\dfrac{dz}{dx}\right)^2 - z\dfrac{dz}{dx} = 0.$

2d Ans. $yz\dfrac{d^2z}{dy^2} + y\left(\dfrac{dz}{dy}\right)^2 - z\dfrac{dz}{dy} = 0.$

10. From $u = xf\left(\frac{y}{x}\right) + \varphi(xy)$ eliminate the functions $f\left(\frac{y}{x}\right)$, $\varphi(xy)$.

$$\text{Ans. } x^2\frac{d^2u}{dx^2} - y^2\frac{d^2u}{dy^2} = 0.$$

11. Eliminate the functions from

$$u = f(x+y) + xy\,\varphi(x-y).$$

$$\text{Ans. } \frac{d^3u}{dx^3} + \frac{d^3u}{dx^2dy} - \frac{d^3u}{dxdy^2} - \frac{d^3u}{dy^3} - \frac{2}{x+y}\left(\frac{d^2u}{dx^2} - \frac{d^2u}{dy^2}\right) = 0.$$

12. From

$$z = f\left(\frac{x}{y}\right) \times \varphi\left(\frac{x^2 - y^2}{x^2 + y^2}\right) \times \psi(xy)$$

eliminate the arbitrary functions f, φ, ψ.

$$\text{Ans. } \left(x^2\frac{d^2z}{dx^2} - y^2\frac{d^2z}{dy^2}\right)z + \left(z - x\frac{dz}{dx} - y\frac{dz}{dy}\right)\left(x\frac{dz}{dx} - y\frac{dz}{dy}\right) = 0.$$

13. From the equations $\frac{d^2x}{dz^2} = \varphi(x, y)$, $\frac{d^2y}{dz^2} = \psi(x, y)$, eliminate the variable z; i.e., change the independent variable from z to x.

$$\text{Ans. } 2\varphi(x, y) = \frac{d}{dx}\frac{\psi(x, y) - \varphi(x, y)\frac{dy}{dx}}{\frac{d^2y}{dx^2}}.$$

14. Eliminate the arbitrary functions from

$$z = x^n f\left(\frac{y}{x}\right) + \frac{1}{y^n}\varphi\left(\frac{x}{y}\right).$$

$$\text{Ans. } x\frac{d^2z}{dx^2} + 2xy\frac{d^2z}{dxdy} + y^2\frac{d^2z}{dy^2} + x\frac{dz}{dx} + y\frac{dz}{dy} - n^2z = 0.$$

DIFFERENTIAL CALCULUS.

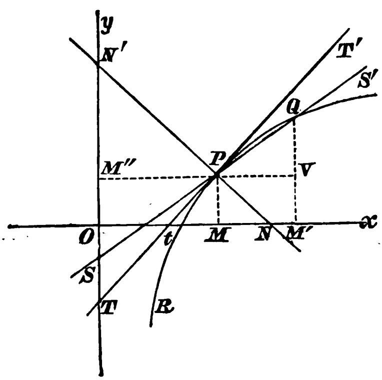
PART SECOND.

GEOMETRICAL APPLICATIONS.

SECTION I.

TANGENTS, NORMALS, SUB-TANGENTS, AND SUB-NORMALS TO PLANE CURVES.

141. ***The tangent line*** to a curve at a given point is the limiting position of a secant line passing through that point, or it is what the secant line becomes when another of its points of intersection with the curve unites with the given point. It is now proposed to find the form of the equation of tangent lines to plane curves.



Let $y = f(x)$ be the equation of the curve RPQ, and take on this curve any point, as P, of which the co-ordinates, referred to the rectangular co-ordinate axes Ox, Oy, are x and y. This point will be briefly designated as point (x, y). Give to x, taken as the independent variable, the increment Δx, y will receive a corresponding increment Δy, and

$x + \Delta x$, $y + \Delta y$, are the co-ordinates of a second point, Q, on the curve; then, if x_1, y_1, denote the general or running co-ordinates of a straight line passing through P and Q, the equation of this line will be

$$y_1 - y = \frac{y + \Delta y - y}{x + \Delta x - x}(x_1 - x),$$

or

$$y_1 - y = \frac{\Delta y}{\Delta x}(x_1 - x).$$

Now, conceive the point Q gradually to approach the point P, $\frac{\Delta y}{\Delta x}$ will, at the same time, gradually approach its limit $\frac{dy}{dx} = y'$, and finally become equal to this limit when Q unites with P; but then the secant line becomes the tangent line. Hence the equation of the tangent line is

$$y_1 - y = \frac{dy}{dx}(x_1 - x), \text{ or } y_1 - y = y'(x_1 - x),$$

in which $\frac{dy}{dx}$ is the tangent of the angle that the tangent line makes with the axis of abscissæ. Calling this angle τ, we have

$$\tan. \tau = \frac{dy}{dx} = y', \ \cot. \tau = \frac{1}{\frac{dy}{dx}} = \frac{1}{y'},$$

$$\cos. \tau = \pm \frac{1}{\sqrt{1 + y'^2}} = \pm \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}},$$

$$\sin. \tau = \pm \frac{y'}{\sqrt{1 + y'^2}} = \pm \frac{\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}.$$

142. *The normal line* to a curve at any point is the straight line passing through the point at right angles to the tangent line at that point.

Since the normal and tangent lines at a given point are perpendicular to each other, denoting the angle that the former makes with the axis of x by ν, we have

$$\tan.\nu = -\frac{1}{\tan.\tau} = -\frac{1}{y'} = -\frac{1}{\frac{dy}{dx}} = -\frac{dx}{dy};$$

and, if x_1, y_1, represent the general co-ordinates of the normal line, its equation is

$$y_1 - y = -\frac{1}{y'}(x_1 - x), \text{ or } y_1 - y = -\frac{dx}{dy}(x_1 - x).$$

Cor. When the equation of the curve is in the form $F(x, y) = 0$, or the ordinate y is an implicit function of the abscissa, we have (Art. 84) $\frac{dy}{dx} = -\frac{\frac{dF}{dx}}{\frac{dF}{dy}}$ hence the equation of the tangent line becomes

$$(x_1 - x)\frac{dF}{dx} + (y_1 - y)\frac{dF}{dy} = 0,$$

and that of the normal

$$(x_1 - x)\frac{dF}{dy} - (y_1 - y)\frac{dF}{dx} = 0.$$

143. To find the equation of the tangent line passing through a given point out of the curve represented by the equation $F(x, y) = 0$, we should make x_1, y_1, in the equation of the tangent, equal to the co-ordinates of the given point. Then, since the point of tangency is common to curve and tangent, the co-ordinates of this point must satisfy both the equation of the curve and the equation of the tangent: hence these two equations will determine x and y, the co-ordinates of the point or points of tangency. In the same way, we may find the equation of a normal line passing through a point not in the curve.

Now, if we have two curves, of which the equations are $F(x,y)=0$, $F(x,y)-c=0$, respectively, the equations of the tangent and of the normal to the first curve will be identically the same as those of the corresponding lines to the second (Art. 142, cor.). Hence, if for given values of x_1, y_1, and any assumed value of c, the values of x and y be deduced from the equations

$$F(x,y)-c=0,\ \ (x_1-x)\frac{dF}{dx}+(y_1-y)\frac{dF}{dy}=0,$$

such values will be the co-ordinates of the points of tangency of the tangent line drawn through the point (x_1, y_1). In like manner, the combination of the equations

$$F(x,y)-c=0,\ (x_1-x)\frac{dF}{dy}-(y_1-y)\frac{dF}{dx}=0,$$

will determine the points of intersection with the curves of the normal lines drawn from the point (x_1, y_1).

Since the equation

$$(x_1-x)\frac{dF}{dx}+(y_1-y)\frac{dF}{dy}=0$$

is independent of c, it will represent a line which is the geometrical locus of the points of tangency of the tangent lines drawn from the point (x_1, y_1), with all the curves which, by ascribing different values to c, can be represented by the equation $F(x,y)-c=0$. So also

$$(x_1-x)\frac{dF}{dy}-(y_1-y)\frac{dF}{dx}=0$$

is the equation of the geometrical locus of the intersections of the normal lines drawn through the point (x_1, y_1) with the same curves. Hence, if these geometrical loci be constructed from their equations, their intersection with the curve answering to an assigned value of c will be the points common to the curve and tangents, or normals, as the case may be.

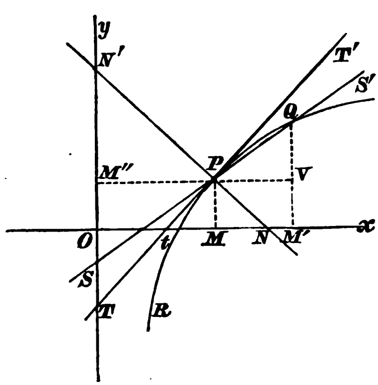
144. Formulas for the distances called ***the tangent, the sub-tangent, the normal,*** and ***the sub-normal.***

Def. 1. ***The tangent*** referred to either axis of co-ordinates is that portion of the tangent line to a curve which is included between the point of tangency and the axis.

Def. 2. ***The sub-tangent*** is that portion of the axis which is included between the intersection of the tangent line with the axis and the foot of that ordinate to the axis, which is drawn from the point of tangency.

Def. 3. ***The normal*** is the part of the normal line included between the point of tangency and the intersection of the normal with the axis.

Def. 4. ***The sub-normal*** is the part of the axis included between the normal and the foot of the ordinate of the point of tangency. The relation of sub-normal to normal is the same as that of sub-tangent to tangent.



In the figure, let P be the point of tangency; then, with reference to the axis of x, PM being the ordinate of P, Pt is the tangent, Mt the sub-tangent, PN the normal, and MN the sub-normal. With reference to the axis of y, the lines of the same name are PT, $M''T$, PN', and $M''N'$, respectively.

$$\text{Now, } \frac{PM}{Mt} = \tan. Ptx = \frac{dy}{dx} : \therefore Mt = MP\frac{1}{\frac{dy}{dx}} = MP\frac{dx}{dy},$$

or $$Mt = sub\text{-}tangent = y\,\frac{dx}{dy}.$$

Again:

$$\frac{MN}{MP} = \tan. MPN = \tan. Ptx = \frac{dy}{dx}:$$

$$\therefore \qquad MN = sub\text{-}normal = y\,\frac{dy}{dx}.$$

Also $\overline{Pt}^2 = \overline{tM}^2 + \overline{PM}^2 = y^2\left(\frac{dx}{dy}\right)^2 + y^2 = y^2\left\{\left(\frac{dx}{dy}\right)^2 + 1\right\}:$

$$\therefore \qquad Pt = tangent = y\sqrt{\left(\frac{dx}{dy}\right)^2 + 1},$$

and $\overline{PN}^2 = \overline{PM}^2 + \overline{MN}^2 = y^2 + y^2\left(\frac{dy}{dx}\right)^2 = y^2\left\{\left(\frac{dy}{dx}\right)^2 + 1\right\}:$

$$\therefore \qquad PN = normal = y\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}.$$

Grouping these formulas, we have

$$Tangent = y\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}. \quad Sub\text{-}tangent = y\,\frac{dx}{dy}.$$

$$Normal = y\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}. \quad Sub\text{-}normal = y\,\frac{dy}{dx}.$$

145. A curve may be given analytically by two equations of the form $y = \varphi(t)$, $x = \psi(t)$, which, by the elimination of t between them, may be reduced to one of the form $y = f(x)$. Without effecting this elimination, the equation of the tangent line will be

$$(y_1 - y)\frac{dx}{dt} - (x_1 - x)\frac{dy}{dt} = 0;$$

and that of the normal,

$$(y_1 - y)\frac{dy}{dt} + (x_1 - x)\frac{dx}{dt} = 0.$$

When the co-ordinate axes are oblique, making with each other an angle ω, the limit of the ratio $\frac{\Delta y}{\Delta x}$ or $\frac{dy}{dx}$ no longer

expresses tan. τ, but $\frac{\sin.\ \tau}{\sin.\ (\omega - \tau)}$. In this case, the investigation and the form of the equation of the tangent line remain unchanged; but the equation of the normal line becomes

$$y_1 - y = \frac{1 + \frac{dy}{dx}\cos.\ \omega}{\frac{dy}{dx} + \cos.\ \omega}(x_1 - x).$$

EXAMPLES.

1. The equation $(x_1 - x)x + (y_1 - y)y = 0$ of the tangent line to the circle can be put under the form

$$\left(x - \frac{x_1}{2}\right)^2 + \left(y - \frac{y_1}{2}\right)^2 = \frac{x_1^2 + y_1^2}{4} \quad (1);$$

which, if x and y are variable, and x_1 and y_1 constant, is the equation of a circle, the centre of which, having $\frac{x_1}{2}$, $\frac{y_1}{2}$, for its co-ordinates, is the middle point of the line drawn from the point (x_1, y_1) to the centre of the given circle. The radius of this circle is equal to $\frac{1}{2}\sqrt{x_1^2 + y_1^2}$. Now, for assigned values of x_1, y_1, the points of contact with the given circumference of the tangent lines drawn from the point (x_1, y_1) must be in the circumferences of both of the circles; and, since (1) is independent of r, the circumference of the circle of which it is the equation is the geometrical locus of all the points of contact with the given circumference of the tangent lines drawn from the point (x_1, y_1) to the different circles that we get by causing r to vary in the equation $x^2 + y^2 = r^2$.

2. The general equation of lines of the second order (conic sections) is

$$u = Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0:$$

$$\therefore \qquad \frac{du}{dx} = By + 2Cx + E, \ \frac{du}{dy} = Bx + 2Ay + D;$$

and the equation of the tangent is

$$(x_1 - x)(By + 2Cx + E) + (y \ - y)(Bx + 2Ay + D) = 0,$$

which the given equation reduces to

$$x_1(By + 2Cx + E) + y_1(Bx + 2Ay + D) + Dy + Ex + 2F = 0.$$

3. ***The logarithmic curve*** is that which has $y = \frac{1}{la} lx$ for its equation. For it we have $\frac{dy}{dx} = \frac{1}{xla}$; and the equations of its tangent and normal lines are

$$xla(y_1 - y) - (x_1 - x) = 0, \ xla(x_1 - x) + (y_1 - y) = 0.$$

The sub-tangent on the axis of y is expressed by $x\frac{dy}{dx} = \frac{1}{la}$, and is therefore constant, and equal to the modulus of the system of logarithms.

4. ***The logarithmic spiral*** is a curve having

$$\frac{y}{x} = \tan. l\frac{\sqrt{x^2 + y^2}}{R}, \text{ or } \tan.^{-1}\frac{y}{x} = l\sqrt{x^2 + y^2} - lR,$$

for its equation; whence

$$\frac{x\frac{dy}{dx} - y}{x^2 + y^2} = \frac{x + y\frac{dy}{dx}}{x^2 + y^2}, \ \frac{dy}{dx} = \frac{x + y}{x - y};$$

and the equations of the tangent and of the normal are

$$(x_1 - x)(x + y) + (y_1 - y)(y - x) = 0,$$

$$(x_1 - x)(y - x) - (y_1 - y)(x + y) = 0.$$

When x_1, y_1, are considered constant, and x, y, are made to vary, these last equations represent two circles, the circumferences of which cut the spiral in the points of contact of the tangents to the spiral which are drawn from the point (x_1, y_1).

5. Denoting the tangent by T, sub-tangent by $T_{,}$, normal

by N, and sub-normal by N_s, determine these lines for the following curves:—

First, The circle: $x^2 + y^2 = r^2$.

$$T = \pm \frac{r}{x}(r^2 - x^2)^{\frac{1}{2}}, \ T_s = \pm \frac{r^2 - x^2}{x}, \ N = r, \ N_s = \pm x.$$

Second, The ellipse or hyperbola: $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \pm 1$.

$$T = \left\{ \left(\frac{2a^4}{x^2} \pm b^2\right)\left(1 \mp \frac{x^2}{a^2}\right) + x^2 - \frac{a^4}{x^2} \right\}^{\frac{1}{2}}, \ T_s = \pm \left(\frac{a^2}{x} \mp x\right).$$

$$N = \frac{b}{a}\left\{ \left(\frac{b^2}{a^2} \mp 1\right) x^2 \pm a^2 \right\}^{\frac{1}{2}}, \ N_s = \pm \frac{b^2}{a^2} x.$$

Third, The parabola: $y^2 = 2px$.

$$T = 2x^{\frac{1}{2}}\left(x + \frac{p}{2}\right)^{\frac{1}{2}}, \ T_s = 2x, \ N = p^{\frac{1}{2}}(p + 2x)^{\frac{1}{2}}, \ N_s = p.$$

The sub-normal in the parabola is constant, and equal to the semi-parameter; the sub-tangent is double the abscissa of the point of tangency.

Fourth, The logarithmic curve: $x = \frac{1}{la} ly$.

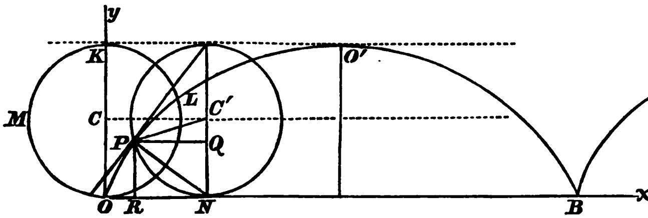
$$T = \sqrt{\left(\frac{1}{la}\right)^2 + e^{2xla}}, \ T_s = \frac{1}{la}, \ N = lae^{2xla}\sqrt{\left(\frac{1}{la}\right)^2 + e^{2xla}},$$
$$N_s = lae^{2xla}.$$

In this curve, the sub-tangent on the axis of x is constant, and equal to the modulus of the system of logarithms.

146. *The Cycloid* is a curve which is generated by a point in the circumference of a circle, while the circle is rolled on a line tangent to its circumference, and kept constantly in the same plane.

Suppose the circle of which C is the centre, and which is tangent to the line Ox at the point O, to roll on this line from

O towards x. While the point of contact is passing from O to N, the radius CO, which, at the origin of the motion, was



perpendicular to Ox, will turn about the centre of the circle through the angle $NC'P$; and the generating point will move from O to P, describing the arc OP of the cycloid. To find the equation of this curve, take Ox, Oy, for the co-ordinate axes. Let $r = CO =$ radius of the generating circle; $\omega = NC'P$ the variable angle; and $x = OR$, $y = PR$, the co-ordinates of the point P: then

$$x = OR = ON - RN = \text{arc } PN - PQ = r\omega - r \sin.\omega = r(\omega - \sin.\omega),$$
$$y = PR = C'N - C'Q = r - r\cos.\omega = r(1 - \cos.\omega).$$

From $y = r(1 - \cos.\omega)$, we have

$$\cos.\omega = \frac{r-y}{r}, \quad \sin.\omega = \pm\frac{1}{r}\sqrt{2ry - y^2}, \quad \omega = \cos.^{-1}\frac{r-y}{r};$$

and these values of ω, $\sin.\omega$, substituted in the equation $x = r(\omega - \sin.\omega)$, give

$$x = r\left(\cos.^{-1}\frac{r-y}{r}\right) \mp \sqrt{2ry - y^2},$$

which is the equation of the cycloid. The minus sign before the radical must be used for points in the arc OPO' which is described while the points in the semi-circumference OLK are brought successively in contact with the line Ox; and the plus sign must be used for points in the arc $O'B$. The point O' is called the *vertex* of the cycloid, or rather the vertex of the branch $OO'B$; since, by continuing the motion of the gen-

erating circle on the indefinite line Ox, we should have an unlimited number of curves in all respects equal to $OO'B$.

From the equation of the cycloid, we get

$$\frac{dx}{dy} = \sqrt{\frac{y}{2r-y}} : \therefore \frac{dy}{dx} = \sqrt{\frac{2r-y}{y}}.$$

Hence the equation of the tangent line at any point is

$$y_1 - y = \sqrt{\frac{2r-y}{y}}(x_1 - x);$$

and of the normal,

$$y_1 - y = -\sqrt{\frac{y}{2r-y}}(x_1 - x).$$

If, in this last equation, we make $y_1 = 0$, we find

$$x_1 - x = \sqrt{y(2r-y)} = r \sin. \omega = RN.$$

Substituting the values of $\frac{dy}{dx}, \frac{dx}{dy}$, in the general formulas, for tangent, sub-tangent, normal, and sub-normal (Art. 144), we have for the cycloid

$$T = y\sqrt{\frac{2r}{2r-y}}, \quad T_s = y\sqrt{\frac{y}{2r-y}},$$

$$N = \sqrt{2ry}, \qquad N_s = \sqrt{y(2r-y)};$$

which last agrees with what was found above; and from which we conclude, that, if supplementary chords be drawn through the extremities of the vertical diameter of the generating circle in any of its positions and the corresponding point of the cycloid, the lower of these chords will be the normal, and the upper the tangent, to the cycloid at that point.

SECTION II.

ASYMPTOTES OF PLANE CURVES.—SINGULAR POINTS.—CONCAVITY AND CONVEXITY.

147. WHEN a plane curve is such, that, as the point of tangency of a tangent line is moved to a greater and greater distance from the origin, the tangent line continually approaches coincidence with a certain fixed line, but cannot be made actually to coincide with it until at least one of the co-ordinates of the point of tangency is made infinite, such fixed line is said to be an *asymptote* to the curve. Hence we may define the asymptote of a curve to be the limiting position of a tangent line when the point of tangency moves to an infinite distance from the origin of co-ordinates.

To establish rules for finding the asymptotes of curves, resume the general equation of a tangent line

$$y_1 - y = \frac{dy}{dx}(x_1 - x) \text{ (Art. 141)},$$

and find from it the expressions for the distances from the origin at which the tangent intersects the co-ordinate axes. These are,

$$x_1 = x - y\frac{dx}{dy} = \text{distance on axis of } x \quad (1),$$

$$y_1 = y - x\frac{dy}{dx} = \text{distance on axis of } y \quad (2).$$

Now, there may be two cases in which asymptotes will exist: 1st, Both $x - y\dfrac{dx}{dy}$, and $y - x\dfrac{dy}{dx}$, may remain finite for the

values $x=\infty$, $y=\infty$. 2d, One of these expressions may remain finite while the other becomes infinite. If the expression for the distance on the axis of x is finite while that for the distance on the axis of y is infinite, the asymptote is parallel to the axis of y; and it is parallel to the axis of x when the distance on the axis of y is finite, and that on the axis of x is infinite.

Ex. 1. The equation of the parabola is

$$y^2=2px;\ \therefore\ \frac{dy}{dx}=\frac{\sqrt{p}}{\sqrt{2x}},\ \frac{dx}{dy}=\frac{\sqrt{2x}}{\sqrt{p}};$$

and, for these values, expressions (1) and (2) for $x=\infty$, $y=\infty$, are both infinite. The parabola, therefore, has no asymptote.

Ex. 2. The equation of the hyperbola is

$$a^2y^2-b^2x^2=-a^2b^2,\ \text{or}\ y=\pm\frac{b}{a}\sqrt{x^2-a^2},$$

$$\frac{dy}{dx}=\pm\frac{bx}{a\sqrt{x^2-a}},\ \text{and}\ x-y\frac{dx}{dy}=x-\frac{x^2-a^2}{x}=\frac{a^2}{x},$$

which reduces to 0 for $x=\infty$: $y-x\dfrac{dy}{dx}$ will also become zero when $x=\infty$, and $\dfrac{dy}{dx}$ becomes $\pm\dfrac{b}{a}$. Hence the hyperbola has two asymptotes passing through its centre, and making equal angles with the transverse axis on opposite sides.

Ex. 3. The exponential curve:

$$y=a^x;\ \therefore\ \frac{dy}{dx}=a^x la,$$

$$x-y\frac{dx}{dy}=x-a^x\frac{1}{a^x la}=x-\frac{1}{la}=\pm\infty\ \text{for}\ x=\pm\infty,$$

$$y-x\frac{dy}{dx}=a^x-xa^x la=\infty\ \text{for}\ x=\infty,\ \text{but}=0\ \text{for}\ x=-\infty;$$

and, $\qquad$ for $x=-\infty$, $\dfrac{dy}{dx}=a^x la=\dfrac{la}{a^\infty}=0.$

Hence the axis of x is an asymptote to the curve, and approaches the curve without limit on the side of x negative. In this reasoning, we have supposed $a > 1$. If $a < 1$, the axis of x is still an asymptote; but, in this case, the curve approaches the axis on the side of x positive.

148. An asymptote to a curve may be defined as the line which the curve continually approaches, but which it can never meet. An investigation, based on this definition, may be given that differs somewhat from the preceding.

Let $y = \alpha x + \beta$ be the equation of a straight line, and $y = \alpha x + \beta + v$ the equation of a curve, v being a function of x and y, which vanishes when x and y are made infinite, or, at least, when one of these variables is made infinite; then the straight line is an asymptote to the curve. For the formula for the perpendicular distance from the point (x, y) to the straight line is $\dfrac{y - \alpha x - \beta}{\sqrt{\alpha^2 + 1}} = \dfrac{v}{\sqrt{\alpha^2 + 1}}$ when the point is a point of the curve. Hence when v vanishes, as it does by hypothesis, for one or both of the values $x = \infty$, $y = \infty$, the straight line is an asymptote to the curve.

From the equation $y = \alpha x + \beta + v$, we have $\dfrac{y}{x} = \alpha + \dfrac{\beta + v}{x}$; whence α is the limit of $\dfrac{y}{x}$ when x and y are increased without limit. In general, for these values of x and y, $\dfrac{y}{x}$ takes the form $\frac{\infty}{\infty}$; but its true value is $\dfrac{\frac{dy}{dx}}{1} = \dfrac{dy}{dx}$. So, also, β is the limit of $y - \alpha x$, and α is the limit of $\dfrac{dy}{dx}$; therefore, in general, β is the limit of $y - \dfrac{dy}{dx}x$.

When the value of α and β thus determined are substituted

in the equation $y = \alpha x + \beta$, it becomes the equation of an asymptote to the curve.

149. When two curves are so related that the difference of the ordinates answering to the same abscissa converges towards zero as the abscissa is increased without limit, or the difference of the abscissæ answering to the same ordinate converges towards zero as the ordinate is increased without limit, either curve is said to be an asymptote to the other.

Suppose we have a curve, the equation of which may be made to take the form

$$y = ax^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n + \frac{b}{x} + \frac{b_1}{x^2} + \frac{b_2}{x^3} + \cdots \quad (1);$$

then the curve represented by

$$y = ax^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \quad (2)$$

will be an asymptote to the first curve.

So also is that represented by

$$y = ax^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n + \frac{b}{x} \quad (3),$$

and

$$y = ax^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n + \frac{b}{x} + \frac{b_1}{x^2} \quad (4).$$

. . . .

It is obvious, also, that of the curves represented by Eqs. 1, 2, 3 . . ., any one is an asymptote to all the others.

Example. Find the asymptotes, rectilinear and curvilinear, of the curve represented by

$$x^3 - xy^2 + ay^2 = 0, \text{ or } y = \pm\sqrt{\frac{x^3}{x-a}}.$$

The value of y may be put under the form $y = \pm x\left(1 - \frac{a}{x}\right)^{-\frac{1}{2}}$; and, expanding this by the Binomial Theorem, we have

$$y = \pm x\left(1 + \frac{a}{2x} + \frac{3a^2}{8x^2} + \frac{5a^3}{16x^3} + \cdots\right) \quad (5),$$

which expresses the true relation between x and y for points of the curve far removed from the origin; for then $\frac{a}{x}$ is less than 1, and the series $1 + \frac{a}{2x} + \frac{3a^2}{8x^2} + \cdots$ converges to a fixed finite limit. Whence we conclude that the curve has two rectilinear asymptotes represented by the equation $y = \pm\left(x + \frac{a}{2}\right)$, and an unlimited number of curves, having for their equations

$$y = \pm\, x\left(1 + \frac{a}{2x} + \frac{3a^2}{8x^2}\right),\ y = \pm\, x\left(1 + \frac{a}{2x} + \frac{3a^2}{8x^2} \cdots\right),$$

which are asymptotes to it and to each other.

150. *Singular points* of curves are those points which offer some peculiarities inherent in the nature of the curve, and independent of the position of the co-ordinate axes.

First, ***Conjugate*** or ***isolated points*** are those the co-ordinates of which satisfy the equation of the curve, but which have no contiguous points in the curve.

Ex. 1. $x^2 + y^2 = 0$ can be satisfied only for $x = 0$, $y = 0$, and represents therefore but a single point; i.e., the origin of co-ordinates.

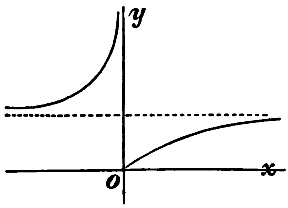
Ex. 2. $y^2 = x^2(x^2 - a^2)$. This is satisfied by $x = 0$, $y = 0$, and therefore the origin belongs to the curve: but there are no points consecutive to it; for values of x between the limits $x = +a$, $x = -a$, make y imaginary. Hence the origin is an isolated point.

Ex. 3. $ay^2 - x^3 + bx^2 = 0$.

Second, ***Points d'arrêt*** are those at which the curves suddenly stop.

Ex. 1. $y = xlx$. Here $x = 0$, $y = 0$, satisfy the equation;

but negative values of x make y imaginary. The origin is therefore a point d'arrêt.



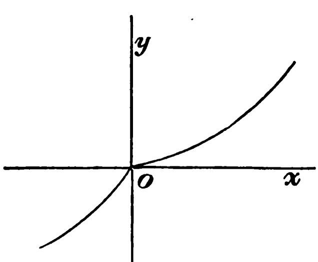
Ex. 2. $y = e^{-\frac{1}{x}}$. If x be indefinitely great, and positive or negative, y approaches the limit 1; but if x be indefinitely small, and positive, y approaches the limit 0; while, for negative and very small values of x, y approaches $+\infty$. The curve will be composed of two branches, as represented in the figure, and will have for the common asymptote to these the parallel to the axis of x at the distance 1.

Third, ***Points saillant*** are those at which two branches of a curve unite and stop, but do not have a common tangent at that point.

Example. $$y = \frac{x}{1 + e^{\frac{1}{x}}}.$$

From this we find

$$\frac{dy}{dx} = \frac{1}{1 + e^{\frac{1}{x}}} + \frac{e^{\frac{1}{x}}}{x\left(1 + e^{\frac{1}{x}}\right)^2}.$$



If x be positive, and be diminished without limit, both y and $\frac{dy}{dx}$ ultimately become zero; but if x be negative, and be numerically diminished without limit, we have ultimately $y = 0$, $\frac{dy}{dx} = 1$. Hence the origin is a point of the curve at which two branches unite having different tangents; one branch having the axis of x for its tangent, and the other a line inclined to the axis of x at an angle of 45°.

Fourth, ***Points de rebroussement,*** or ***cusps,*** are points at which two branches of a curve meet a common tangent, and stop at that point. The cusp is of the *first species* if the two branches lie on opposite sides of the tangent, and of the *second species* if the branches lie on the same side of the tangent.

Fifth, ***Multiple points*** are points at which two or more branches of a curve meet, but do not all stop, or at which at least three branches meet and stop.

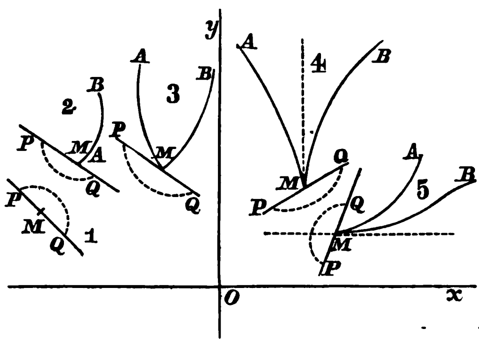
Ex. 1. $y^2 = x^2(1 - x^2)$ represents a curve of two branches which cross at the origin, at which the equations of the tangents are $y = -x$, $y = x$.

Ex. 2. The equation $y^2 = x^4(1 - x^2)$ is that of a curve composed of two branches which meet at the origin, and have the axis of x for a common tangent. The origin is a multiple point.

Sixth, ***A point of inflexion*** is one at which the curve and its tangent at that point cross each other.

151. We will now establish the analytical conditions by which the existence and nature of singular points in a curve, if it have any, may be generally recognized; omitting, for the present, the case in which the first differential co-efficient of the ordinate of the curve becomes infinite.

If a curve has either a conjugate point, a point d'arrêt, a point saillant, or a cusp of the first or second species, we may pass through this point an indefinite number of straight lines, such that, in the vicinity of this point, there is not on one side of any one of these lines for the last three kinds of points just named, or on either side for that first named, any point belonging to the curve under consideration.



This is illustrated in the adjoining figure, in which M, Fig. 1, is a conjugate point; M, Fig. 2, is a point d'arrêt; M, Fig. 3, a point saillant; and M, Figs. 4 and 5, are cusps of the first and second species.

Now, if, for any one of these cases, two points, P, Q, be taken on one of these lines, one on each side of the point M, and however near to it, these points may be united by a curve which has no point in common with the given curve AB. Consequently, if $u = f(x, y) = 0$ is the equation of AB, and u is continuous, as is supposed, it cannot change sign, except at zero: but no values of x, y, belonging to PQ, can reduce u to zero; for, if so, then that point would be common to AB and PQ. Hence the values of x, y, belonging to points of PQ, make the sign of u constant; while the values of x, y, belonging to the point M, reduce u to zero.

Since, then, the value of u at the point M is zero, and has the same sign at P, on one side of this point, that it has at Q on the other, these points being very near M, u must be a maximum or minimum at M according as the sign of u at P and Q is negative or positive. In either case, we must have

$$\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} = 0.$$

Again: denoting the tangent of the angle that the arbitrary straight line PMQ makes with the axis of x by a, the equation of this line, which the co-ordinates of M must satisfy, will be

$y = ax + b$: whence $\frac{dy}{dx} = a$; and, substituting this above, we have

$$\frac{du}{dx} + a\frac{du}{dy} = 0.$$

But this last equation must hold for an indefinite number of values for a, since the line PMQ is arbitrary; and therefore we must have

$$\frac{du}{dx} = 0, \ \frac{du}{dy} = 0.$$

The co-ordinates of the four kinds of singular points under consideration should then satisfy, at the same time, the three equations

$$u = 0, \ \frac{du}{dx} = 0, \ \frac{du}{dy} = 0.$$

Two of these equations will determine values of x and y to substitute in the third. If a set of these values $x = x_0$, $y = y_0$, verifies the three equations, the corresponding point may be a singular point, but not necessarily so.

To ascertain the nature of the point thus determined, let us seek the value of $\frac{dy}{dx}$, which the equation $\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} = 0$ gives under the form $\frac{0}{0}$. The second differential equation, because of the conditions $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$, reduces to

$$\frac{d^2u}{dy^2}\left(\frac{dy}{dx}\right)^2 + 2\frac{d^2u}{dxdy}\frac{dy}{dx} + \frac{d^2u}{dx^2} = 0 \quad (a).$$

Suppose, also, that, by the solution of $u = f(x, y) = 0$, we have found $y = F(x)$ for the equation of the branch of the

curve on which the point about which we are inquiring is situated. The solution of Eq. (a) with respect to $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = -\frac{\frac{d^2u}{dxdy} \mp \sqrt{\left(\frac{d^2u}{dxdy}\right)^2 - \frac{d^2u}{dx^2}\frac{d^2u}{dy^2}}}{\frac{d^2u}{dy^2}}.$$

Hence,

I. From the definition of a conjugate point and these equations, we conclude that the point $x = x_0$, $y = y_0$, will be conjugate: *first*, if the two ordinates

$$y_1 = F(x_0 + h),\ y_2 = F(x_0 - h),$$

are both imaginary; *second*, if the curve at this point has no tangent, which requires that

$$\left(\frac{d^2u}{dxdy}\right)^2 - \frac{d^2u}{dx^2}\frac{d^2u}{dy^2} < 0;$$

unless we have

$$\frac{d^2u}{dx^2} = 0,\ \frac{d^2u}{dxdy} = 0,\ \frac{d^2u}{dy^2} = 0.$$

II. The point $x = x_0$, $y = y_0$, will be a point d'arrêt: *first*, when only one of the ordinates $y = F(x_0 + h)$, $y = F(x_0 - h)$, is imaginary; *second* if the curve at this point has but one tangent, which will be the case when the co-ordinates of the point satisfy the equation $\frac{d^2u}{dy^2} = 0$.

III. The point $x = x_0$, $y = y_0$, will be a point saillant: *first*, if to each of the abscissæ $x = x_0 + h$, $x = x_0 - h$, there is but one corresponding ordinate, differing but little from y_0, or if there are two, and but two ordinates, differing but little from y_0, corresponding to one of these abscissæ, and none to the other abscissa; *second*, if the curve at the point x_0, y_0, has two tangents, which requires that we have

$$\left(\frac{d^2u}{dxdy}\right)^2 - \frac{d^2u}{dx^2}\frac{d^2u}{dy^2} > 0.$$

IV. The point x_0, y_0, will be a cusp, when, the first condition for a point saillant being fulfilled, the two tangents at that point coincide; which cannot be the case unless

$$\left(\frac{d^2u}{dxdy}\right)^2 - \frac{d^2u}{dx^2}\frac{d^2u}{dy^2} = 0.$$

152. To investigate the conditions for multiple points, let the equation $F(x, y) = 0$ in rational form represent the curve; then

$$\frac{dF}{dx} + \frac{dF}{dy}\frac{dy}{dx} = 0 \quad \text{(Art. 84)}.$$

Since at least two branches of a curve pass through a multiple point, two or more tangents may be drawn at that point: hence $\frac{dy}{dx}$, for such a point, must have more than one value. But, since $F(x, y)$ is supposed rational, $\frac{dF}{dx}$, $\frac{dF}{dy}$, will each admit of but one value for the values of x_0, y_0, which determine the point. Therefore $\frac{dy}{dx}$ cannot have more than one value, unless $\frac{dF}{dx} = 0$, $\frac{dF}{dy} = 0$; and these are the conditions for the existence of a multiple point. The equation from which to find the values of $\frac{dy}{dx}$ is

$$\frac{d^2F}{dx^2} + 2\frac{d^2F}{dxdy}\frac{dy}{dx} + \frac{d^2F}{dy^2}\left(\frac{dy}{dx}\right)^2 = 0 \quad (b),$$

which will give two real values for $\frac{dy}{dx}$, if, for the values of x_0 and y_0,

$$\left(\frac{d^2F}{dxdy}\right)^2 - \frac{d^2F}{dx^2}\frac{d^2F}{dy^2} > 0;$$

and in this case the multiple point is called a *double point*.

If $$\frac{d^2F}{dx^2}=0,\ \frac{d^2F}{dxdy}=0,\ \frac{d^2F}{dy^2}=0,$$

then Eq. (b) becomes indeterminate, and we must pass to the differential equation of the third order, which, after introducing the above conditions, i.e. $\frac{dF}{dx}=0,\ \frac{d^2F}{dx^2}=0\ldots$, is

$$\frac{d^3F}{dx^3}+3\frac{d^3F}{dx^2dy}\frac{dy}{dx}+3\frac{d^3F}{dxdy^2}\left(\frac{dy}{dx}\right)^2+\frac{d^3F}{dy^3}\left(\frac{dy}{dx}\right)^3=0 \quad (d).$$

This cubic equation will give three values for $\frac{dy}{dx}$, which, if all real, show that three tangents can be drawn to the curve at the point (x_0, y_0): the point is then called a *triple point*. If Eq. (d) becomes indeterminate, we proceed to the differential equation of the fourth order, and thus get an equation of the fourth degree for finding $\frac{dy}{dx}$; and, in general, if n branches of a curve unite in a multiple point, the co-ordinates of such point must verify the following equations:

$$\frac{dF}{dx}=0,\ \frac{dF}{dy}=0,\ \frac{d^2F}{dx^2}=0,\ \frac{d^2F}{dxdy}=0,\ \frac{d^2F}{dy^2}=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^{n-1}F}{dx^{n-1}}=0,\ \frac{d^{n-1}F}{dx^{n-2}dy}=0\ldots,\ \frac{d^{n-1}F}{dy^{n-1}}=0;$$

and the n^{th} differential equation of the curve would in general determine n real values for $\frac{dy}{dx}$.

153. If a curve has a point of inflexion, the co-ordinates of that point must verify the equation $\frac{d^2y}{dx^2}=0$.

Suppose the equation of the curve has been put under the form $y=F(x)$; then the difference Δy of the ordinates corresponding to the abscissæ x and $x+h$ is (Art. 61)

$$\Delta y=hF'(x)+\frac{h^2}{1.2}F''(x)+\cdots+\frac{h^n}{1.2\ldots n}F^{(n)}(x+\theta h).$$

The difference of the ordinates corresponding to the same abscissæ of the tangent line at the point (x, y) is $\Delta y_1 = hF'(x)$: hence, denoting $\Delta y - \Delta y_1$ by δ, we have

$$\delta = \frac{h^2}{1.2} F''(x) + \frac{h^3}{1.2.3} F'''(x) + \cdots + \frac{h^n}{1.2\ldots n} F^{(n)}(x + \theta h).$$

When h is very small, the first term in the expression for δ exceeds the sum of all the others; and consequently the sign of δ for points in the vicinity of the point (x, y) will be constantly positive, or constantly negative, according as $F''(x)$ is positive or negative: hence, if $F''(x)$ does not vanish, the curve cannot cross the tangent at the point (x, y), and there can be no point of inflexion. If $F''(x)$ vanishes, then the first term in the value of δ is $\frac{h^3}{1.2.3} F'''(x)$, if $F'''(x)$ does not vanish at the same time; and the sign of this term will change from positive to negative, or the reverse, as h changes from positive to negative. This can only be the case when the curve crosses the tangent at the point (x, y); and this point is therefore a point of inflexion. If $F'''(x) = 0$, then, by the same course of reasoning, we prove that the co-ordinates of a point of inflexion must verify the equation $F''''(x) = 0$, &c. Thus, to find the co-ordinates of a point of inflexion, we seek the roots common to the equations

$$y = F(x), \quad F''(x) = 0, \quad \text{or } f(x, y) = 0, \ \frac{d^2 y}{dx} = 0.$$

A system $x = x_0$, $y = y_0$, of these roots, will be the co-ordinates of such a point, if the first of the derivatives that does not vanish for them is of an odd order.

154. Throughout this investigation of the conditions for singular points, we have supposed $F(x)$, and its derivatives for values of x and y in the vicinity of those corresponding to

the point (x_0, y_0), to be continuous. But, if $\frac{dy}{dx} = \infty$, we may readily determine the nature of the point (x_0, y_0). Under this hypothesis, the two quantities $F(x_0 + h)$, $F(x_0 - h)$, may both be real; or one may be real, and the other imaginary.

First, If both are real, and both greater or both less than $F(x_0)$, the point (x_0, y_0) will be a cusp of the first species: if one is greater and the other less than $F(x_0)$, the point will be a point of inflexion.

Second, If one of these quantities, say $F(x_0 - h)$, is real, and the other imaginary, then, if $F(x_0 - h)$ has but one value, the point will be a point d'arrêt: if $F(x_0 - h)$ has two values, both of which are greater or both less than $F(x_0)$, the point will be a cusp of the second species; but, if one of these values is greater and the other less than $F(x_0)$, the point will be simply a limit of the curve.

Third, If each, or but one, of the quantities

$$F(x_0 + h), \ F(x_0 - h),$$

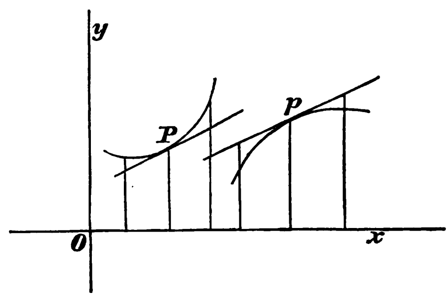
has more than two values, the point (x_0, y_0) will be, in general, both a multiple point and a point of inflexion.

In conclusion, to obtain the co-ordinates of singular points of curves, we seek the values of x and y that will reduce the differential co-efficients to zero, to infinity, or to $\frac{0}{0}$. The nature of the point is ascertained by inquiring how many branches of the curve pass through the point, and determining the position of the tangent line or tangent lines corresponding to the point.

155. The terms "concave" and "convex" are employed to express the sense or direction in which, starting from a given point, the curve bends with reference to a given line

from the tangent at that point. If it bends from the tangent towards the line, it is said to be concave, or to have its concavity turned towards the line; but, if the sense in which it bends from the tangent is from the line, it is said to be convex, or to have its convexity turned towards the line.

To find the conditions of the concavity or convexity of a curve towards a given line, take that line for the axis of x, and let P, of which the co-ordinates are x and y, be the point at which the curve is to be examined with reference to these properties. Draw the tangent at P: then, from our definition, if at P the curve be convex to the axis of x, the ordinates of the curve for the abscissæ $x+h$, $x-h$, must be greater than the corresponding ordinates of the tangent at P; h having any value between some small but finite limit and zero. But, if the curve be concave towards the axis of x, the reverse must be the case.



If the equation of the curve is $y = F(x)$, the ordinate corresponding to the abscissa $x+h$ is

$$y+\Delta y = F(x) + hF'(x) + \frac{h^2}{1.2}F''(x) + \cdots$$
$$+ \frac{h^n}{1.2\ldots n}F^{(n)}(x+\theta h).$$

The equation of the tangent to the curve at the point (x, y) is $y_1 - y = F'(x)(x_1 - x)$, or $y_1 = F(x) + x_1F'(x) - xF'(x)$. Observing that x, y, are the co-ordinates of the point of tan-

gency, the ordinate of the tangent corresponding to the abscissa $x + h$ is

$$y_1 + \Delta y_1 = F(x) + xF'(x) + hF'(x) - xF'(x)$$
$$= F(x) + hF'(x):$$

hence, if δ denote the difference $y + \Delta y - (y_1 + \Delta y_1)$, we have

$$\delta = \frac{h^2}{1.2} F''(x) + \cdots + \frac{h^n}{1.2\ldots n} F^{(n)}(x + \theta h).$$

The sign of this difference, when h is very small, is the same as that of $\frac{h^2}{1.2} F''(x)$, which has the sign of $F''(x)$ whether h be positive or negative: therefore, if $F''(x)$ be positive, the curve is convex to the axis of x; and it is concave if $F''(x)$ be negative.

We have supposed the point of the curve at which its convexity or concavity was examined to be above the axis of x, or to have a positive ordinate. Had the point been below the axis, $F''(x)$ positive would have indicated concavity, and $F''(x)$ negative would have indicated convexity. To include both cases in one enunciation, we say, "When a curve at any point is convex to the axis of x, $y\frac{d^2y}{dx^2}$ is positive at that point; when it is concave to the axis of x, $y\frac{d^2y}{dx^2}$ is negative."

Cor. Comparing this article with Art. 153, we conclude, that, when a curve has a point of inflexion, it will be convex to a given line on one side of the point of inflexion, and concave on the other.

EXAMPLES.

Find the asymptotes to the curves represented by the following equations: —

1. $y^3 = x^2(2a - x)$. Ans. $y = -x + \frac{2a}{3}$.

2. $y^3 = (x - a)^2(x - c)$. Ans. $y = x - \frac{1}{3}(2a + c)$.

3. $x^2y^2 = a^2(x^2 - y^2)$. Ans. $y = \pm a$.

4. $(y - 2x)(y^2 - x^2) - a(y - x)^2 + 4a^2(x + y) = a^3$.

Ans. $y = x,\ y = -x + \frac{2a}{3},\ y = 2x + \frac{a}{3}$.

Find and describe the singular points in the curves of which the following are the equations: —

5. $y = \frac{x^3}{a^2 + x^2}$. There is a point of inflexion at the origin, and also at the point having $x = \pm a\sqrt{3}$ for its abscissa.

6. $y(a^4 - b^4) = x(x - a)^4 - xb^4$. There are two points of inflexion corresponding to the abscissæ $x = a$, $x = \frac{2a}{5}$.

7. $y^3 = (x - a)^2(x - c)$. There is a cusp of the first species at the point of which $x = a$ is the abscissa.

8. $x^4 - ax^2y - axy^2 + a^2y^2 = 0$. There is a conjugate point at the origin.

9. $ay^2 - x^3 + bx^2 = 0$. There is a conjugate point at the origin, and a point of inflexion at the point having $x = \frac{4b}{3}$ for its abscissa.

SECTION III.

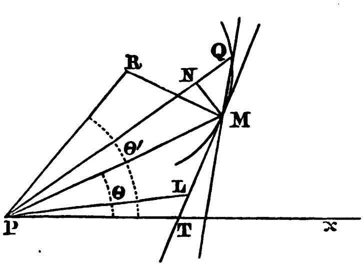
POLAR CO-ORDINATES.—DIFFERENTIAL CO-EFFICIENTS OF THE ARCS AND AREAS OF PLANE CURVES.—OF SOLIDS AND SURFACES OF REVOLUTION.

156. LET the pole coincide with the origin of a system of rectangular co-ordinate axes: denote the radius vector by r, and the angle, called *vectorial angle*, that it makes with the axis of x taken as the *initial line*, or *polar axis*, by θ; then the formulas by which an equation expressed in terms of rectangular co-ordinates may be transformed into one expressed in terms of polar co-ordinates are $x = r\cos.\theta$, $y = r\sin.\theta$.

To express in polar co-ordinates the tangent of the angle that a tangent line to a curve makes with the axis of x, we have, calling this angle τ, $\tan.\tau = \frac{dy}{dx}$; and hence (Eqs. a, Art. 132)

$$\tan.\tau = \frac{\sin.\theta\frac{dr}{d\theta} + r\cos.\theta}{\cos.\theta\frac{dr}{d\theta} - r\sin.\theta};$$

and from this we may readily find the expression for the tangent of the angle that the tangent line at any point makes with the radius vector of that point.



Let M be the point, P the pole, MT the tangent line, and Px the axis of x, from which θ is estimated; then

$$PMT = MTx - MPT:$$

hence, by the formula for the tangent of the difference of two arcs,

$$\tan. PMT = \frac{\dfrac{\sin.\theta\dfrac{dr}{d\theta} + r\cos.\theta}{\cos.\theta\dfrac{dr}{d\theta} - r\sin.\theta} - \tan.\theta}{1 + \dfrac{\tan.\theta\left(\sin.\theta\dfrac{dr}{d\theta} + r\cos.\theta\right)}{\cos.\theta\dfrac{dr}{d\theta} - r\sin.\theta}} = r\frac{d\theta}{dr}.$$

This may also be found directly as follows: Take on the curve a second point, Q, the co-ordinates of which are $r + \Delta r$, $\theta + \Delta\theta$, and draw MN perpendicular to PQ; then $MN = r\sin.\Delta\theta$, and $QN = r + \Delta r - r\cos.\Delta\theta$: hence

$$\tan. NQM = \frac{r\sin.\Delta\theta}{r + \Delta r - r\cos.\Delta\theta}.$$

Now let the point Q move towards M. The limiting position of the secant QM is the tangent MT, and the limit of the angle NQM is the angle PMT. Call this angle β; then

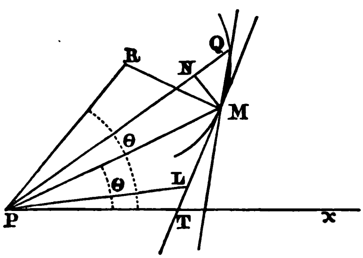
$$\tan.\beta = \lim.\frac{r\sin.\Delta\theta}{r + \Delta r - r\cos.\Delta\theta} = \lim.\frac{r\sin.\Delta\theta}{2r\sin.^2\dfrac{\Delta\theta}{2} + \Delta r}$$

$$= \lim.\frac{\dfrac{r\sin.\Delta\theta}{\Delta\theta}}{\dfrac{2r\sin.^2\dfrac{\Delta\theta}{2}}{\Delta\theta} + \dfrac{\Delta r}{\Delta\theta}}.$$

The limit of $\dfrac{\sin.\Delta\theta}{\Delta\theta} = 1$, $\lim.\dfrac{\sin.^2\dfrac{\Delta\theta}{2}}{\Delta\theta} = \lim.\dfrac{\sin.\dfrac{\Delta\theta}{2}}{\Delta\theta}\sin.\dfrac{\Delta\theta}{2} = 0$,

and $\lim.\dfrac{\Delta r}{\Delta\theta}$ is denoted by $\dfrac{dr}{d\theta}$: therefore $\tan.\beta = r\dfrac{d\theta}{dr}$.

157. To find the polar equations of the tangent and normal lines to a curve, we may assume the equations of these lines referred to rectangular axes (Arts. 141, 142), and change them into their equivalents in polar co-ordinates; or we may proceed thus: —



Let r and θ be the co-ordinates of the point M; and r', θ', those of a second point, L, in the tangent line: then from the triangle PLM, making

$$PML = \tau,$$

we have

$$\frac{r}{r'} = \frac{\sin. PLM}{\sin. PML} = \frac{\sin.(\theta - \theta' + \tau)}{\sin. \tau}$$

$$= \sin.(\theta - \theta')\cot.\tau + \cos.(\theta - \theta'),$$

or
$$\frac{r}{r'} = \frac{1}{r}\frac{dr}{d\theta}\sin.(\theta - \theta') + \cos.(\theta - \theta') \quad (1),$$

observing that $\cot.\tau = \frac{1}{\tan.\tau} = \frac{1}{r}\frac{dr}{d\theta}$ (Art. 156). Eq. 1 may be written,

$$r^2 = r'\frac{d}{d\theta}r\sin.(\theta - \theta') \quad (2).$$

Making $u = \frac{1}{r}$, $u' = \frac{1}{r'}$; then $-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{du}{d\theta}$: and hence, by dividing both members of (1) by r, and substituting these values, we find

$$u' = u\cos.(\theta - \theta') - \frac{du}{d\theta}\sin.(\theta - \theta') \quad (3).$$

To find the polar equation of the normal at any point of a curve, denote by r and θ the co-ordinates of M; and by r', θ', those of any point, R, in the normal: then

$$\frac{PM}{PR} = \frac{\sin. PRM}{\sin. PMR} = \frac{\sin.\left(\theta' - \theta + \frac{\pi}{2} - \tau\right)}{\sin.\left(\frac{\pi}{2} - \tau\right)}:$$

therefore $\quad \frac{r}{r'} = \sin. (\theta' - \theta) \tan. \tau + \cos. (\theta' - \theta)$

$$= \sin. (\theta' - \theta) \frac{r d\theta}{dr} + \cos. (\theta' - \theta) \quad (4),$$

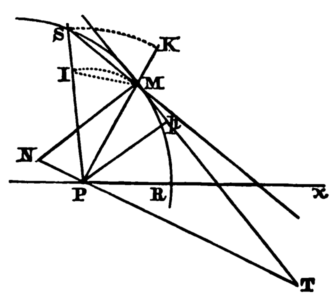
which may be written

$$r' \frac{d}{d\theta} r \cos. (\theta - \theta') = r \frac{dr}{d\theta}.$$

Adopting a notation like that in the case of the tangent, (4) becomes

$$u' = u \cos. (\theta' - \theta) - u^2 \frac{d\theta}{du} \sin. (\theta' - \theta) \quad (5).$$

158. Let P be the polar point to which is referred the curve RMS, and through P draw NT perpendicular to the radius vector PM; then MT being the tangent, and MN the normal, to the curve at the point M, the lines MT, PT, MN, and PN, are, respectively, the polar tangent, sub-tangent, normal, and sub-normal.



To find the formulas for the lengths of these lines, put angle $PMT = \beta$, and resume the equation $\tan. \beta = r \frac{d\theta}{dr}$ (Art. 156), making $\frac{d\theta}{dr} = \frac{1}{r'}$: whence $\tan. \beta = \frac{r}{r'}$, from which we find

$$\cos. \beta = \frac{r'}{\sqrt{r^2 + r'^2}}, \quad \sin. \beta = \frac{r}{\sqrt{r^2 + r'^2}}.$$

Then the triangle PMT gives

$$MT = T = \frac{MP}{\cos. PMT} = \frac{r}{\cos. \beta} = \frac{r}{r'}\sqrt{r^2 + r'^2} = r\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2},$$

$$PT = T_s = PM \tan. PMT = r\frac{r}{r'} = \pm r^2\frac{d\theta}{dr},$$

$$MN = N = \frac{PM}{\cos. PMN} = \frac{PM}{\sin. PMT} = \sqrt{r^2 + r'^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2},$$

$$PN = N_s = PM \tan. PMN = r \cot. \beta = r\frac{r'}{r} = r' = \pm\frac{dr}{d\theta}.$$

The polar sub-tangent is considered positive when it is on the right, and negative when on the left, of the line *PM*; the eye being supposed at *P*, and looking from *P* toward *M*. The sign of the sub-tangent will then be the same as that of $\frac{dr}{d\theta}$; that is, positive when r is an increasing function of θ, and negative when r is a decreasing function of θ.

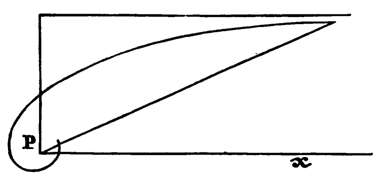
159. An asymptote to a curve referred to polar co-ordinates is a tangent line, the polar sub-tangent to which remains finite when the radius vector of the point of tangency becomes infinite. Hence, to find the asymptotes to a polar curve, we must seek the values of θ, which make r infinite while $r^2\frac{d\theta}{dr}$ remains finite. If m be a value of θ which satisfies these conditions, the asymptote may be constructed by drawing through the pole a line, making, with the initial line, the angle m, and another line at right angles to this through the same point; laying off on the latter, to the right or to the left according as $r^2\frac{d\theta}{dr}$ is positive or negative, the distance represented by $r^2\frac{d\theta}{dr}$, and through the extremity of this distance drawing a line parallel to the first line. The line last drawn will be the asymptote.

Example. In the hyperbolic spiral, so called because of the similarity of its equation $r=\frac{a}{\theta}$, or $r\theta=a$, to that of the hyperbola referred to its asymptotes, we have

$$\frac{d\theta}{dr}=-\frac{\theta^2}{a}: \therefore r^2\frac{d\theta}{dr}=-\frac{a^2}{\theta^2}\times\frac{\theta^2}{a}=-a.$$

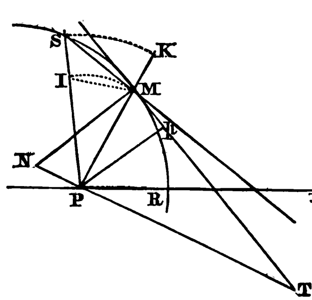
Hence the sub-tangent is constant, and equal to $-a$: but $\theta=0$ gives $r=\infty$; whence the line parallel to the polar axis at the distance from it equal to $-a$ is an asymptote to the curve.

This curve, beginning at an infinite distance, continually approaches the pole, making an indefinite number of turns around without ever reaching it.



160. When the curve in the vicinity of a tangent line at any point, and the pole, lie on the same side of the tangent, the curve at that point is concave to the pole; but, if the curve and the pole lie on opposite sides of the tangent, the curve in the vicinity of the point of tangency is convex to the pole.

Let $Pp=p$ be a perpendicular from the pole on the tangent to the curve at the point (θ, r): then it is plain, that, if the curve at this point is concave to the pole, p will increase or decrease as r increases or decreases; that is, p is an increasing function of r; and, on the contrary, if the curve is convex to the pole, p is a decreasing function of r. Hence, when the curve is concave to the



pole, $\frac{dp}{dr}$ must be positive; and, when convex, $\frac{dp}{dr}$ must be negative (Art. 51).

Cor. At a point of inflexion, the curve with reference to the pole must change from concave to convex, or the reverse. Hence, for a point of inflexion, $\frac{dp}{dr} = 0$ or ∞.

We have (Art. 158)

$$\sin. PMT = \sin. \beta = \frac{r}{\sqrt{r^2 + r'^2}} = \frac{r}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}};$$

$$\therefore \quad p = r \sin. \beta = \frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}, \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

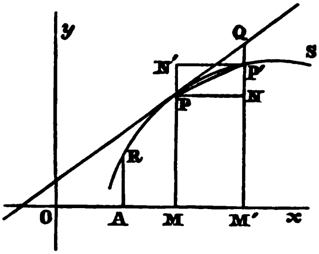
Make $\quad u = \frac{1}{r}$; then $\frac{du}{d\theta} = -\frac{1}{r^2} \cdot \frac{dr}{d\theta}, \quad \left(\frac{dr}{d\theta}\right)^2 = r^4 \left(\frac{du}{d\theta}\right)^2$:

$$\therefore \quad \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2, \text{ and } -\frac{1}{p^3}\frac{dp}{d\theta} = \left(u + \frac{d^2u}{d\theta^2}\right)\frac{du}{d\theta};$$

$$\therefore \quad \frac{dp}{d\theta}\frac{d\theta}{du} = \frac{dp}{du} = -p^3\left(u + \frac{d^2u}{d\theta^2}\right):$$

hence $$\frac{dp}{dr} = \frac{dp}{du}\frac{du}{dr} = -\frac{1}{r^2}\frac{dp}{du} = \frac{p^3}{r^2}\left(u + \frac{d^2u}{d\theta^2}\right).$$

Therefore, at a point of inflexion, $u + \frac{d^2u}{d\theta^2}$ will generally change its sign.



161. Differential co-efficient of the arc of a plane curve.

Let $F(x, y) = 0$ or $y = f(x)$ be the equation of the curve *RPS* referred to the rectangular axes *Ox*, *Oy*; and take, in this curve, any point, *P*, of which the co-ordinates are x, y. Denote the length of the curve estimated

from a fixed point to the point P by s; then, if x be increased by $MM' = \Delta x$, s is increased by the arc $PP' = \Delta s$, and it is required to find the limit of the ratio $\frac{\Delta s}{\Delta x}$, or the differential co-efficient of the arc s, regarded as a function of x.

The tangent line to the curve at the point P meets the ordinate $P'M'$, produced, if necessary, at Q; and makes, with the axis of x, an angle of which the tangent, sine, and cosine are respectively

$$y', \quad \pm\frac{y'}{\sqrt{1+y'^2}}, \quad \frac{1}{\sqrt{1+y'^2}}.$$

Now, if, within the interval Δx, the curve is continually concave or continually convex to the chord PP', it is evident that arc $PP' >$ chord PP', and arc $PP' < PQ + QP'$.

But chord $PP' = \sqrt{\Delta x^2 + \Delta y^2}$, $PQ = \frac{PN}{\cos. QPN} = \Delta x\sqrt{1+y'^2}$,

$$QN = PN \tan. QPN = y'\Delta x; \quad \therefore \quad QP' = y'\Delta x - \Delta y:$$

hence, substituting in the preceding inequalities, we have

$$\Delta s > \sqrt{\Delta x^2 + \Delta y^2}, \quad \Delta s < \Delta x\sqrt{1+y'^2} + y'\Delta x - \Delta y.$$

Therefore

$$\frac{\Delta s}{\Delta x} > \sqrt{1 + \frac{\Delta y^2}{\Delta x^2}}, \quad \frac{\Delta s}{\Delta x} < \sqrt{1+y'^2} + y' - \frac{\Delta y}{\Delta x}.$$

At the limit, the second member of each of these inequalities reduces to $\sqrt{1+y'^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$: hence we have

$$\lim. \frac{\Delta s}{\Delta x} = \frac{ds}{dx} = \sqrt{1+y'^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

for the differential co-efficient of the arc regarded as a function of the abscissa. This must be understood as expressing only the absolute value of $\frac{ds}{dx}$; for the arc may be an increas-

ing or a decreasing function of the abscissa, according as it is estimated in the direction of x positive or x negative: hence the general value should be written

$$\frac{ds}{dx} = \pm\sqrt{1+y'^2} = \pm\sqrt{1+\left(\frac{dy}{dx}\right)^2};$$

the sign $+$ to be taken if s increases with x, and the sign $-$ in the opposite case.

Cor. 1. Since

$$\lim.\frac{PQ+QP'}{PP'} = \lim.\frac{\Delta x\sqrt{1+y'^2}+y'\Delta x-\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$= \lim.\frac{\sqrt{1+y'^2}+y'-\frac{\Delta y}{\Delta x}}{\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}} = 1,$$

and the arc PP' is always included between the chord PP' and the broken line $PQ+QP'$, it follows that $\lim.\frac{\Delta s}{\sqrt{\Delta x^2+\Delta y^2}}=1$; and hence, when the arc is infinitely small, it and its chord become equal.

Cor. 2. Squaring both members of the equation

$$\frac{ds}{dx} = \pm\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}},$$

and multiplying through by $\left(\frac{dx}{ds}\right)^2$, we find

$$1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2.$$

Now, if x and y are both functions of a third variable z, then

$$\frac{dx}{dz} = \frac{dx}{ds}\,\frac{ds}{dz}; \quad \therefore \frac{dx}{ds} = \frac{\frac{dx}{dz}}{\frac{ds}{dz}}.$$

$$\frac{dy}{dz} = \frac{dy}{ds}\frac{ds}{dz}: \therefore \frac{dy}{ds} = \frac{\frac{dy}{dz}}{\frac{ds}{dz}}.$$

These values of $\frac{dx}{ds}$, $\frac{dy}{ds}$, substituted in the preceding equation, give

$$\frac{ds}{dz} = \pm \left\{ \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 \right\}^{\frac{1}{2}}.$$

162. Denote by α and β the angles that the tangent line at the point P (figure of last article) makes with the axes of x and y positive; then

$$\cos.\alpha = \lim. \frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} = \frac{1}{\sqrt{1 + y'^2}} = \frac{dx}{ds} \quad \text{(Arts. 41, 161)},$$

$$\cos.\beta = \lim. \frac{\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} = \frac{dy}{dx}\frac{dx}{ds} = \frac{dy}{ds} \quad \text{(Art. 42)};$$

or, more generally, writing the sign $\pm$ before $\sqrt{\Delta x^2 + \Delta y^2}$,

$$\cos.\alpha = \pm\frac{dx}{ds}, \quad \cos.\beta = \pm\frac{dy}{ds},$$

in which the upper sign or the lower sign is to be used according as the angle is that made with the axis by the tangent produced from the point P in the direction in which the arc increases from that point, or the opposite. This refers to the algebraic signs of $\cos.\alpha$, $\cos.\beta$: their essential signs are determined by combining their algebraic signs with the essential signs of $\frac{dx}{ds}$, $\frac{dy}{ds}$.

163. Differential co-efficient of an arc referred to polar co-ordinates.

For the transformation of rectangular into polar co-ordinates,

we have $x = r\cos.\theta$, $y = r\sin.\theta$. We also have (Arts. 42, 161)

$$\frac{ds}{d\theta} = \frac{ds}{dx}\frac{dx}{d\theta} = \frac{dx}{d\theta}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}.$$

But $\dfrac{dx}{d\theta} = \cos.\theta\dfrac{dr}{d\theta} - r\sin.\theta$, $\dfrac{dy}{d\theta} = \sin.\theta\dfrac{dr}{d\theta} + r\cos.\theta$;

therefore $\dfrac{ds}{d\theta} = \sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}$: and, in like manner,

$$\frac{ds}{dr} = \frac{ds}{d\theta}\frac{d\theta}{dr} = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}.$$

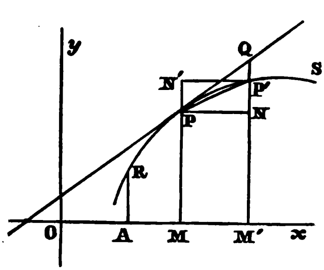
Cor. When β is the angle included between the radius vector of a curve at the point (r, θ) and the tangent line at that point, we have (Art. 156) $\tan.\beta = r\dfrac{d\theta}{dr}$; and hence

$$\sin.\beta = \frac{r\dfrac{d\theta}{dr}}{\sqrt{1+r^2\left(\dfrac{d\theta}{dr}\right)^2}} = \frac{r\dfrac{d\theta}{dr}}{\dfrac{ds}{dr}} = r\frac{d\theta}{ds},$$

and

$$\cos.\beta = \frac{1}{\sqrt{1+r^2\left(\dfrac{d\theta}{dr}\right)^2}} = \frac{1}{\dfrac{ds}{dr}} = \frac{dr}{ds}.$$

164. Differential co-efficient of the area of a plane curve.



The area enclosed by the arc RP of a plane curve, a given ordinate AR, the ordinate of any point P of the curve, and the axis Ox, is obviously a function of the abscissa of P, since the area varies with the position of this point on the curve.

Let $ARPM = u$, and give to x, the abscissa of P, the increment

$MM' = \Delta x$; then $MPP'M' = \Delta u$ is the corresponding increment of u, and we are to find the limit of the ratio $\frac{\Delta u}{\Delta x}$.

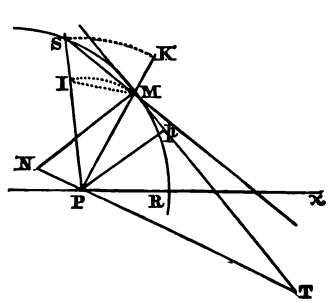
Through P and P' draw parallels to Ox, and limited by the ordinates PM, $P'M'$; and suppose Δx to be so small, that, between these ordinates, the ordinate of the curve constantly increases or constantly decreases. The rectilinear areas $MPP'M'$, $MN'P'M'$, have $y\Delta x$, $(y + \Delta y)\,\Delta x$, for their respective measures, and the curvilinear area $MPP'M'$ is constantly included between these two; that is,

$$\Delta u > y\,\Delta x, \quad \Delta u < (y + \Delta y)\,\Delta x:$$

whence $y < \frac{\Delta u}{\Delta x} < y + \Delta y$; or, by passing to the limit,

$$\frac{du}{dx} = y, \quad du = ydx.$$

If the co-ordinate axes are oblique, making with each other the angle ω, the above demonstration still applies, observing that then the area Δu lies between the areas of two parallelograms, the sides of which are parallel to the axes; and, since the area of a parallelogram is measured by the product of its adjacent sides multiplied by the sine of the included angle, we should have $\frac{du}{dx} = y \sin.\ \omega$.



165. When the curve is referred to polar co-ordinates, the area considered is a sector embraced by a given radius vector PR, the radius vector PM of any point (θ, r), and the arc RM. Denote this area, which is a function of θ, by u; let θ be increased by $\Delta\theta$, by which the point M moves to S, and u is increased by

the sector $PMS = \Delta u$; and suppose $\Delta\theta$ so small, that the radius vector of the arc from M to S is constantly increasing or constantly decreasing. With P as a centre, and the radii vectores $PM = r$, $PS = r'$, as radii, describe the arcs MI, SK, limited by these radii vectores. We have

$$\text{sector } PMI < \Delta u < \text{sector } PSK;$$

or, since sector $PMI = \frac{1}{2}r^2\Delta\theta$, and sector $PSK = \frac{1}{2}r'^2\Delta\theta$,

$$\frac{1}{2}r^2\Delta\theta < \Delta u < \frac{1}{2}r'^2\Delta\theta: \therefore \frac{1}{2}r^2 < \frac{\Delta u}{\Delta\theta} < \frac{1}{2}r'^2.$$

But, at the limit, r' becomes equal to r: hence

$$\frac{du}{d\theta} = \frac{1}{2}r^2, \quad du = \frac{1}{2}r^2 d\theta.$$

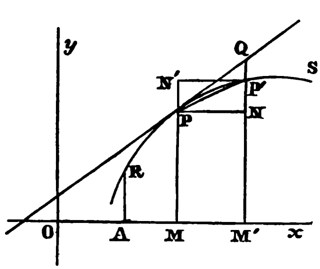
The equations $x = r\cos.\theta$, $y = r\sin.\theta$, give $\frac{y}{x} = \tan.\theta$: whence, by differentiating with respect to θ,

$$x\frac{dy}{d\theta} - y\frac{dx}{d\theta} = \frac{x^2}{\cos.^2\theta} = \frac{r^2\cos.^2\theta}{\cos.^2\theta} = r^2,$$

$$\frac{1}{2}(xdy - ydx) = \frac{1}{2}r^2 d\theta;$$

an expression in terms of rectangular co-ordinates for the differential of a polar area that is of frequent use.

166. Differential co-efficient of the volume of a solid of revolution.



If V represent the volume generated by the revolution of the plane area $ARPM$ about the axis Ox, and x be increased by $MM' = \Delta x$, the corresponding increment ΔV of V will be the volume generated in the revolution by the area $MPP'M'$. Now, if Δx be so small, that y increases constantly from P to P', ΔV will

be included between the volumes of the cylinders generated by the rectangles

$$MPNM', \ MN'P'M'.$$

Hence, denoting MP by y, and $M'P'$ by y_1, we have

$$\pi y^2 \Delta x < \Delta V < \pi y_1^2 \Delta x, \text{ or } \pi y^2 < \frac{\Delta V}{\Delta x} < \pi y_1^2;$$

that is, $\frac{\Delta V}{\Delta x}$ is comprised between the two quantities πy^2, πy_1^2, the second of which converges to equality with the first as Δx diminishes. Hence, at the limit,

$$\frac{dV}{dx} = \pi y^2, \ dV = \pi y^2 dx.$$

167. Differential co-efficient of the surface of a solid of revolution.

Let s represent the arc RP (figure of last article), and S the surface generated by the revolution of this arc about the axis Ox. For the increment $MM' = \Delta x$ of x, s will be increased by the arc $PP' = \Delta s$; and S, by $\Delta S =$ the surface generated by Δs. When Δx is sufficiently small, the surface ΔS will be comprised between the surface of the conical frustum generated by the chord PP', and the surface generated by the broken line PQP'. The surfaces generated by the chord and by the broken line are measured by

$$\pi(2y + \Delta y)\sqrt{\Delta x^2 + \Delta y^2},$$

$$\pi\left(2y + \frac{dy}{dx}\Delta x\right)\sqrt{(\Delta x)^2 + (\Delta y)^2} + 2\pi y\frac{dy}{dx}\Delta x + \pi\left(\frac{dy}{dx}\Delta x\right)^2$$
$$- 2\pi y \Delta y - \pi(\Delta y)^2:$$

hence, establishing the inequalities, and dividing through by Δx, we have

$$\frac{\Delta S}{\Delta x} > \pi(2y + \Delta y)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2},$$

$$\frac{\Delta S}{\Delta x} < \pi\left(2y + \frac{dy}{dx}\Delta x\right)\sqrt{1 + \left(\frac{dy}{dx}\right)^2} + \pi\left(\frac{dy}{dx}\right)^2\Delta x - \pi\left(\frac{\Delta y}{\Delta x}\right)^2\Delta x$$
$$+ 2\pi y\frac{dy}{dx} - 2\pi y\frac{\Delta y}{\Delta x}.$$

At the limit, the second member of each of these inequalities becomes equal to $2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$: hence

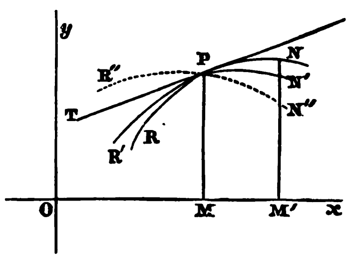
$$\lim. \frac{\Delta S}{\Delta x} = \frac{dS}{dx} = 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 2\pi y\frac{ds}{dx},$$

$$dS = 2\pi y ds.$$

SECTION IV.

DIFFERENT ORDERS OF CONTACT OF PLANE CURVES. — OSCULATORY CURVES. — OSCULATORY CIRCLE. — RADIUS OF CURVATURE. — EVOLUTES, INVOLUTES, AND ENVELOPES.

168. SUPPOSE $y = F(x)$, $y = f(x)$, to be the equations of the two curves RPN, $R'PN'$, which have a common point P; and let us compare the ordinates $M'N$, $M'N'$, of these curves corresponding to the same abscissa $OM' = x + h$, differing but little from the abscissa $OM = x$ of the point P. We have



$$M'N = F(x + h), \ M'N' = f(x + h):$$

$$\therefore \qquad NN' = F(x + h) - f(x + h).$$

Developing each term in the value of NN' by the formula of Art. 61, observing that $F(x) = f(x)$ by hypothesis, we find

$$NN' = h\left(\frac{dF}{dx} - \frac{df}{dx}\right) + \frac{h^2}{1.2}\left(\frac{d^2F}{dx^2} - \frac{d^2f}{dx^2}\right) + \cdots$$

$$+ \frac{h^n}{1.2\ldots n}\left(\frac{d^nF}{dx^n} - \frac{d^nf}{dx^n}\right) + \frac{h^{n+1}}{1.2\ldots n+1}\left(\frac{d^{n+1}F}{dx^{n+1}} - \frac{d^{n+1}f}{dx^{n+1}}\right)$$

$$+ \frac{h^{n+2}}{1.2\ldots n+2}\left(F^{(n+2)}(x + \theta h) - f^{(n+2)}(x + \theta_1 h)\right),$$

the last term of which may be written, —

$$\frac{h^{n+1}}{1.2\ldots n+1}\,\frac{h}{n+2}\left(F^{(n+2)}(x+\theta h)-f^{(n+2)}(x+\theta_1 h)\right)$$
$$=\frac{h^{n+1}}{1.2\ldots n+1}R;$$

R being a quantity that vanishes with h: hence

$$NN'=h\left(\frac{dF}{dx}-\frac{df}{dx}\right)+\frac{h^2}{1.2}\left(\frac{d^2F}{dx^2}-\frac{d^2f}{dx^2}\right)+\cdots$$
$$+\frac{h^{n+1}}{1.2\ldots n+1}\left(\frac{d^{n+1}F}{dx^{n+1}}-\frac{d^{n+1}f}{dx^{n+1}}+R\right).$$

If, in addition to $F(x)=f(x)$, we have $\frac{dF}{dx}=\frac{df}{dx}$, the curves have a common tangent, PT, at the point P, and are said to have a contact of the *first order*: and if, at the same time, $\frac{d^2F}{dx^2}=\frac{d^2f}{dx^2}$, the contact is of the *second order;* and, generally, the contact is of the n^{th} *order* if n denotes the highest order of the differential co-efficients of the ordinates of the two curves that become equal when in them the co-ordinates of the common point are substituted.

169. When two curves have a contact of the n^{th} order, no third curve can pass between them in the vicinity of their common point, unless it have, with each of the two curves, a contact of an order at least equal to the n^{th}. For $y=F(x)$, $y=f(x)$, being the equations of two curves, RPN, $R'PN'$ (figure of the last article), which have at the point P a contact of the n^{th} order, let $y=\varphi(x)$ be the equation of a third curve, $R''PN''$, passing through P, and having with the first curve a contact of the m^{th} order, m being less than n. Then, by the preceding article, we should have

$$F(x)=\varphi(x),\ \frac{dF}{dx}=\frac{d\varphi}{dx}\ldots,\ \frac{d^mF}{dx^m}=\frac{d^m\varphi}{dx^m},$$

and $$NN'' = \frac{h^{m+1}}{1.2\ldots(m+1)}\left(\frac{d^{m+1}F}{dx^{m+1}} - \frac{d^{m+1}\varphi}{dx^{m+1}} + R_1\right) \quad (a),$$

also $$NN' = \frac{h^{n+1}}{1.2\ldots(n+1)}\left(\frac{d^{n+1}F}{dx^{n+1}} - \frac{d^{n+1}f}{dx^{n+1}} + R\right) \quad (b);$$

R and R_1 being quantities which vanish with h: hence

$$\frac{NN'}{NN''} = \frac{h^{n-m}}{(m+2)\ldots(n+1)} \times \frac{\dfrac{d^{n+1}F}{dx^{n+1}} - \dfrac{d^{n+1}f}{dx^{n+1}} + R}{\dfrac{d^{m+1}F}{dx^{m+1}} - \dfrac{d^{m+1}\varphi}{dx^{m+1}} + R_1}.$$

Since, as h converges towards the limit 0, R and R_1 converge towards the same limit, and reach it at the same time h does, and since $n > m$, it follows that the ratio $\dfrac{NN'}{NN''}$ can be made as small as we please by giving to h a sufficiently small value; that is, when h is a very small quantity, NN' will be less than NN'', and the curve $y = \varphi(x)$ cannot, in the vicinity of the common point P, pass between the curves $y = F(x)$, $y = f(x)$.

It is evident that this reasoning holds when h is negative as well as when it is positive.

Cor. When h is sufficiently small, the sign of the expression for NN' (Eq. b) will be the same as that of h^{n+1}, and will therefore change with that of h if n be even, but remain invariable if n be odd. Hence, if two curves have a contact of an even order, they will cross each other at the point of contact, but not otherwise.

170. Osculatory curves. If the form of the function $F(x)$, and the constants which enter it, are given, the equation $y = F(x)$ represents a curve fully determined in respect to species, magnitude, and position; but if the form of the function only is known, the constants which enter it being arbitrary, the species of curve is all that the equation determines. Thus the equation $y = b \pm \sqrt{r^2 - (x-a)^2}$, when r, a, b, are fixed in

value, represents a circle that is completely known; but if r, a, and b are undetermined, the equation may represent every possible circle lying in the plane of the co-ordinate axes. It is then the equation of the species "circle."

When a curve of a given species has a higher order of contact than any other curve of that species with a given curve, the former is said to be an *osculatrix* to the latter.

Suppose $f(x_1, y_1, a, b, c\ldots) = 0$ (1), involving $n+1$ arbitrary constants, to be the equation of the species of curve that is to be made an osculatrix to the curve of which $y = F(x)$ (2) is the equation. By means of the $n+1$ constants in (1), we can satisfy the $n+1$ equations

$$y = y_1, \frac{dy}{dx} = \frac{dy_1}{dx}, \frac{d^2y}{dx^2} = \frac{d^2y_1}{dx^2} \ldots, \frac{d^ny}{dx^n} = \frac{d^ny_1}{dx^n} \quad (3);$$

or, in other words, these equations will determine the values of $a, b, c\ldots$, which, substituted in (1), will make it the equation of a curve having a contact of the n^{th} order with the curve represented by (2); and it will be an osculatrix, since, in general, a higher order of contact cannot be imposed. We conclude from the above, that the number denoting the order of contact of an osculatory curve is one less than the number of constants entering the equation of the curve.

Example. The form of the equation of a straight line is $y = ax + b$; and since this equation contains two constants, a and b, we may so determine them as to cause the line to have a contact of the first order with a given curve at a given point. Suppose $y = F(x)$ to be the equation of the curve, and that $x = m$, $y = n$, are the co-ordinates of the point; then the equations to be satisfied are

$$am + b = F(m), \quad a = F'(m),$$

which determine a and b.

171. Osculatory circle, or ***circle of curvature.*** Assume the co-ordinate axes to be rectangular, and let $y = F(x)$ (1) be the equation of the given curve; then, since $(x_1 - a)^2 + (y_1 - b)^2 = \rho^2$ (2) is the general equation of the circle, and contains three constants, the osculatory circle will have, with the given curve, a contact of the second order.

From (2) we get, by two successive differentiations,

$$\left.\begin{aligned} x_1 - a + (y_1 - b)\frac{dy_1}{dx_1} = 0 \\ 1 + \left(\frac{dy_1}{dx_1}\right)^2 + (y_1 - b)\frac{d^2y_1}{dx_1^2} = 0 \end{aligned}\right\} \quad (3);$$

and, because the circle is to be the circle of curvature, we must have

$$y = y_1,\ \frac{dy}{dx} = \frac{dy_1}{dx_1},\ \frac{d^2y}{dx^2} = \frac{d^2y_1}{dx_1^2} \quad (4).$$

These values of y_1, $\frac{dy_1}{dx_1}$, $\frac{d^2y_1}{dx_1^2}$, substituted in Eqs. 3, give

$$x - a + (y - b)\frac{dy}{dx} = 0,\ 1 + \left(\frac{dy}{dx}\right)^2 + (y - b)\frac{d^2y}{dx^2} = 0 \quad (5):$$

therefore

$$y - b = -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}},\ x - a = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}} \quad (6).$$

By substituting these values of $y - b$, $x - a$, for $y_1 - b$, $x_1 - a$, respectively, in Eq. 2, we find

$$\rho = \pm\frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad (7).$$

Eqs. 6 will determine the position of the centre; and Eq. 7, the length of the radius of the osculatory circle to the given curve at any point. When the curve at the point of

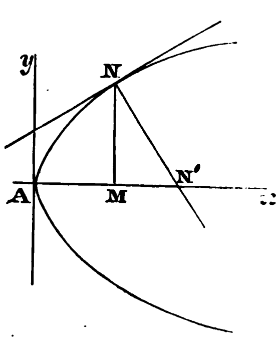
osculation is concave to the axis of x, as is the case if y is positive and $\frac{d^2y}{dx^2}$ negative, then, to make ρ positive, we must take the minus sign written before the second member of (7).

The first of Eqs. 5 indicates that the centre of the circle is in the normal to the curve at the point of osculation; and from the second of these equations we conclude that $y - b$ and $\frac{d^2y}{dx^2}$ must have opposite signs, and hence that the centre of the circle is always on the concave side of the curve, since $y - b$ is the difference between the ordinate of the point of contact and the ordinate of the centre of the osculatory circle.

In general, the contact of an osculatory circle is of the second order, that is, of an even order; and consequently it crosses the curve at the point of contact, except at particular points where the contact is of an order higher than the second.

The osculatory circle is often called *circle of curvature;* and its centre and radius, the *centre* and *radius of curvature.*

172. As an application of the formulas of the preceding article, let it be required to find the radius of curvature of a conic section at any point of the curve. If the curve be referred to one of its axes, and to the tangent through its vertex, as co-ordinate axes, its equation will be



$$y^2 = 2px + qx^2,$$

which, by two differentiations, gives

$$\frac{dy}{dx} = \frac{p + qx}{y}, \quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = q.$$

In the last of these, substituting for $\frac{dy}{dx}$ its value taken from the first, we have, —

$$y\frac{d^2y}{dx^2}+\frac{p^2+2pqx+q^2x^2}{y^2}=q:$$

whence

$$\frac{d^2y}{dx^2}=-\frac{p^2}{y^3};$$

and for ρ we have

$$\rho=\frac{y^3\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{p^2}.$$

The numerator of this value of ρ is the cube of the normal NN'; for from the triangle MNN' we have

$$\overline{NN'}^2=n^2=y^2+y^2\frac{dy^2}{dx^2}: \quad \therefore \quad n^3=y^3\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}},$$

and $$\rho=\frac{n^3}{p^2}.$$

Therefore the radius of curvature at any point of a conic section is equal to the cube of the normal at that point divided by the square of the semi-parameter.

The value of ρ expressed in terms of the constants of the equation, and the abscissa of the point N, is

$$\rho=\frac{\left((q+q^2)x^2+2p(1+q)x+p^2\right)^{\frac{3}{2}}}{p^2}.$$

173. The equation of the tangent line to a curve at the point (x, y) being

$$y-y_1=\frac{dy}{dx}(x-x_1),$$

the expression for the length of the perpendicular p let fall from the origin of co-ordinates on the tangent is

$$p=\frac{x\frac{dy}{dx}-y}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}};$$

whence, by differentiation and reduction,

$$\frac{dp}{dx}=\frac{x\frac{d^2y}{dx^2}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}-\frac{dy}{dx}\frac{d^2y}{dx^2}\left(x\frac{dy}{dx}-y\right)}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

$$=\frac{\left(x+y\frac{dy}{dx}\right)\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}=\frac{1}{\rho}\left(x+y\frac{dy}{dx}\right) \quad \text{(Art. 171).}$$

And, if r be the distance from the origin to the point of tangency,

$$r^2=x^2+y^2;\ \therefore\ r\frac{dr}{dx}=x+y\frac{dy}{dx}:$$

and, substituting this value of $x+y\frac{dy}{dx}$ in the expression for $\frac{dp}{dx}$, we have

$$\frac{dp}{dx}=\frac{1}{\rho}r\frac{dr}{dx};\ \therefore\ \rho=r\frac{dr}{dp}.$$

174. If x and y are both functions of a third variable, s, then

$$\frac{dy}{dx}=\frac{\frac{dy}{ds}}{\frac{dx}{ds}},\quad \frac{d^2y}{dx^2}=\frac{\frac{d^2y}{ds^2}\frac{dx}{ds}-\frac{d^2x}{ds^2}\frac{dy}{ds}}{\left(\frac{dx}{ds}\right)^3} \quad (1);\ \text{(Art. 129)};$$

and these values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, put in the formula for ρ (Art. 171), give

$$\rho=\frac{\left\{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{ds^2}\frac{dx}{ds}-\frac{d^2x}{ds^2}\frac{dy}{ds}} \quad (2).$$

Supposing s to be the arc of the curve estimated from a fixed

point, we find, from the formula $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ of Art. 161,

$$\frac{dx}{ds} = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}; \quad \therefore \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{dx}\right)^2 \left(\frac{dx}{ds}\right)^2 = 1;$$

$$\therefore \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \quad (3),$$

$$\rho = \frac{1}{\frac{d^2y}{ds^2}\frac{dx}{ds} - \frac{d^2x}{ds^2}\frac{dy}{ds}}, \quad \frac{1}{\rho} = \frac{d^2y}{ds^2}\frac{dx}{ds} - \frac{d^2x}{ds^2}\frac{dy}{ds} \quad (4).$$

From (3), by differentiating, we get

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} = 0 \quad (5).$$

Squaring (4) and (5), and adding, we find

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

Eliminating $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$, in turn, between (4) and (5), observing that

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1,$$

we also find

$$\frac{1}{\rho} = \frac{\frac{d^2y}{ds^2}}{\frac{dx}{ds}} = -\frac{\frac{d^2x}{ds^2}}{\frac{dy}{ds}}$$

175. To find the expression for the radius of curvature in terms of the polar co-ordinates of the curve, we substitute in the value of ρ, Art. 171, the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, given in Art. 131, thus getting

$$\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}};$$

and, when $r = \frac{1}{u}$, we have

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}, \quad \frac{d^2r}{d\theta^2} = \frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2 - \frac{1}{u^2}\frac{d^2u}{d\theta^2};$$

and these values, substituted in the above value of ρ, give

$$\rho = \frac{\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{u^2\left(u + \frac{d^2u}{d\theta^2}\right)}.$$

176. The chord of curvature at any point of a curve is the portion of a secant line through that point that is included between the point and the arc of the circle of curvature at the same point.

The chord of curvature that, produced if necessary, passes through the pole, is obtained by multiplying 2ρ by the cosine of the angle included between the radius vector and the normal to the curve at the point; but if r is the radius vector, and p the perpendicular let fall from the pole on the tangent to the curve, $\frac{p}{r}$ is the cosine of the angle included between the radius vector and the normal. But the value of p is readily found to be $\frac{1}{\sqrt{u^2 + \left(\frac{du}{d\theta}\right)^2}}$: hence the chord of curvature through the pole is equal to

$$2\rho\frac{p}{r} = 2p\frac{dr}{dp} = \frac{2u^2 + 2\left(\frac{du}{d\theta}\right)^2}{u^2\left(u + \frac{d^2u}{d\theta^2}\right)} \quad \text{(Arts. 173, 175).}$$

177. Denoting by α the angle which the tangent to a curve at any point makes with the axis of abscissæ, we have

$$\tan. \alpha = \frac{dy}{dx}, \quad \alpha = \tan.^{-1} \frac{dy}{dx}:$$

therefore

$$\frac{d\alpha}{ds} = \frac{\frac{d^2 y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} \frac{dx}{ds} = \frac{\frac{d^2 y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}},$$

since $\dfrac{dx}{ds} = \dfrac{1}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}}$: therefore $\rho = \dfrac{ds}{d\alpha}$.

178. The co-ordinates of the points of the curve at which the radius of curvature is a maximum or a minimum must be found from the equation of the curve and the equation $\dfrac{d\rho}{dx} = 0$; the latter leading to

$$3 \left(\frac{d^2 y}{dx^2}\right)^2 \frac{dy}{dx} - \frac{d^3 y}{dx^3} \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = 0 \quad (1).$$

Differentiating the second of Eqs. 3, Art. 171, we find

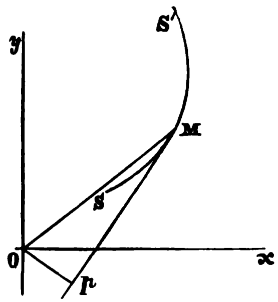
$$3 \frac{dy_1}{dx_1} \frac{d^2 y_1}{dx_1^2} + (y_1 - b) \frac{d^3 y_1}{dx_1^3} = 0:$$

$$\therefore \frac{d^3 y_1}{dx_1^3} = - \frac{3 \frac{dy_1}{dx_1} \frac{d^2 y_1}{dx_1^2}}{y_1 - b} = \frac{3 \left(\frac{d^2 y}{dx^2}\right)^2 \frac{dy}{dx}}{1 + \left(\frac{dy}{dx}\right)^2} \quad (2)$$

by Eqs. 4 of the same article.

Comparing Eqs. 1 and 2, it is seen that $\dfrac{d^3 y_1}{dx_1^3} = \dfrac{d^3 y}{dx^3}$; which proves, that, at the points of maximum or minimum cur-

vature, the osculatory circle has, with the given curve, a contact of the third order.



179. If a perpendicular be let fall from the origin of co-ordinates on the tangents drawn to the different points of any curve, as SMS', the locus of the intersections of the perpendiculars with the tangents will be a new curve, the properties of which will depend on those of the given curve.

Denote the co-ordinates of the new curve by x_1, y_1; then will the length of the perpendicular p_1, from the origin to the tangent drawn to this curve at the point corresponding to the point (x, y) of the given curve, have for its expression

$$p_1 = \frac{x_1 \frac{dy_1}{dx_1} - y_1}{\sqrt{1 + \left(\frac{dy_1}{dx_1}\right)^2}}.$$

The equation of the tangent to the given curve is

$$\mu - y = \frac{dy}{dx}(\nu - x),$$

μ and ν being the general co-ordinates. Since the point (x_1, y_1) is on this tangent,

$$y_1 - y = \frac{dy}{dx}(x_1 - x).$$

The equation of Op is $\mu = \frac{y_1}{x_1}\nu$; and, because Op is perpendicular to Mp, $\frac{y_1}{x_1} = -\frac{dx}{dy}$: whence

$$(y_1 - y)y_1 = -x_1(x_1 - x),$$

or

$$yy_1 + xx_1 = x_1^2 + y_1^2.$$

Differentiating this last with respect to x, we find

$$y_1\frac{dy}{dx}+y\frac{dy_1}{dx}+x_1+x\frac{dx_1}{dx}=2y_1\frac{dy_1}{dx}+2x_1\frac{dx_1}{dx}.$$

Substituting for $\frac{dy}{dx}$ its value, $-\frac{x_1}{y_1}$, transposing and reducing, we have

$$\frac{dy_1}{dx_1}=-\frac{2x_1-x}{2y_1-y};$$

and, by means of this, p_1 becomes

$$p_1=\frac{x_1^2+y_1^2}{\sqrt{x^2+y^2}}=\frac{p^2}{r};$$

r being the distance from the origin to the point (x, y) of the given curve, and p the perpendicular Op let fall from the origin on the tangent to the curve at the same point.

180. If $f(x, y)=0$ be the equation of a curve, it has been shown (Art. 171), that, calling μ, ν, the co-ordinates of the centre of curvature corresponding to the point (x, y) of the given curve, we have

$$x-\mu+(y-\nu)\frac{dy}{dx}=0 \quad (1),$$

$$1+\left(\frac{dy}{dx}\right)^2+(y-\nu)\frac{d^2y}{dx^2}=0 \quad (2).$$

By means of these equations, the equation of the curve, and its first and second differential equations, we may eliminate x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and find a direct relation between μ and ν. This will be the equation of a new curve, called, with reference to the given curve, *the evolute;* the given curve being *the involute.*

It is evident that μ and ν may be considered as functions of

x; and, if Eq. 1 be differentiated under this supposition, we have

$$1+\left(\frac{dy}{dx}\right)^2+(y-\nu)\frac{d^2y}{dx^2}-\frac{d\mu}{dx}-\frac{d\nu}{dx}\frac{dy}{dx}=0;$$

and, through (2), this reduces to

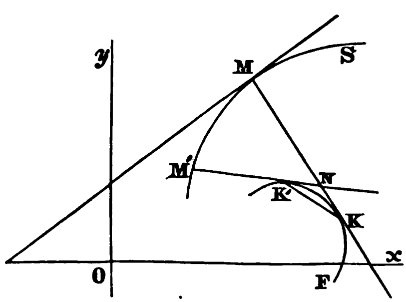
$$\frac{d\mu}{dx}+\frac{d\nu}{dx}\frac{dy}{dx}=0; \therefore\ 1+\frac{d\nu}{d\mu}\frac{dy}{dx}=0:$$

whence, by the substitution of the value of $\frac{dy}{dx}$ derived from the last of these equations, Eq. 1 becomes

$$y-\nu=\frac{d\nu}{d\mu}(x-\mu).$$

These relations show that the tangent to the evolute is a normal to the corresponding point of the involute, and the converse.

A consequence of this property is, that the evolute of a curve is the locus of the intersections of the consecutive normals to this curve. For take the two normals MK, $M'K'$, which, by what precedes, are tangent to the evolute at the points K, K'. When the point M' is made to approach the point M, the line $M'K'$ approaches the line MK, and the points K' and N tend to unite in the point K: hence the point K may be regarded as the intersection of the normal MK with the normal indefinitely near or consecutive to it.



Another important consequence is, that the length of the arc of the evolute between two centres of curvature is the

difference of the corresponding radii of curvature. To prove this, differentiate the equation

$$\rho^2 = (x-\mu)^2 + (y-\nu)^2,$$

treating y, μ, ν, and ρ as functions of x: we thus have

$$\rho\frac{d\rho}{dx} = (x-\mu)\left(1-\frac{d\mu}{dx}\right) + (y-\nu)\left(\frac{dy}{dx}-\frac{d\nu}{dx}\right),$$

which, by Eq. 1, reduces to

$$\rho\frac{d\rho}{dx} = -(x-\mu)\frac{d\mu}{dx} - (y-\nu)\frac{d\nu}{dx} \quad (a).$$

From Eq. 1 and the equation $\frac{d\mu}{dx} + \frac{d\nu}{dx}\frac{dy}{dx} = 0$, we get

$$\frac{\frac{d\mu}{dx}}{x-\mu} = \frac{\frac{d\nu}{dx}}{y-\nu} = \pm\left\{\frac{\left(\frac{d\mu}{dx}\right)^2 + \left(\frac{d\nu}{dx}\right)^2}{(x-\mu)^2+(y-\nu)^2}\right\}^{\frac{1}{2}} = \pm\frac{1}{\rho}\frac{ds}{dx} \quad (b)$$

when s denotes the length of the arc of the evolute estimated from any point F (*Cor.* 2, Art. 161): whence

$$\frac{(x-\mu)\frac{d\mu}{dx}}{(x-\mu)^2} = \frac{(y-\nu)\frac{d\nu}{dx}}{(y-\nu)^2} = \pm\frac{1}{\rho}\frac{ds}{dx} \quad (c).$$

And, by the combination of Eqs. a and c, we find $\frac{ds}{dx} = \pm\frac{d\rho}{dx}$: wherefore, since $\frac{d(s\mp\rho)}{dx} = 0$, it follows that $s\mp\rho$ is equal to some constant which we will denote by l; that is,

$$s\mp\rho = l,\ s'\mp\rho' = l;\ \therefore\ s-s' = \rho-\rho',$$

or $$\text{arc } FK' - \text{arc } FK = MK - M'K' = \text{arc } KK'.$$

Suppose a flexible but inextensible string, of a length equal to $M'K'$ plus the arc $K'KF$, fastened by one of its ends to F, to envelope the curve FKK', and then pass in the direction of the tangent to the curve at K' from K' to M'. If this string be unwound from this curve, its free end will describe the

curve $MM'S$. It is from this property that the terms "evolute" and "involute" are derived. It is also seen that there may be an unlimited number of involutes answering to the same evolute FKK', and that, to describe them, it is only necessary to lengthen or shorten arbitrarily the part of the string that extends in the tangent to the evolute. Since the tangents to the evolute are normals to all these involutes, it follows that the latter curves have the same normals and the same centres of curvature, and that the parts of the common normals included between any two will be equal: hence one involute enables us to find all the others.

181. Radius of curvature and evolute of the ellipse.

The equation of an ellipse, referred to its centre and axes, is

$$a^2y^2 + b^2x^2 = a^2b^2:$$

whence
$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \quad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}.$$

These values, substituted in the formula for the radius of curvature, Art. 171, give

$$\rho = \frac{\left(1 + \dfrac{b^4x^2}{a^4y^2}\right)^{\frac{3}{2}}}{\dfrac{b^4}{a^2y^3}} = \frac{(b^4x^2 + a^4y^2)^{\frac{3}{2}}}{a^4b^4}.$$

To find the equation of the evolute of the ellipse, resume the equations

$$x - \mu + (y - \nu)\frac{dy}{dx} = 0 \quad (1),$$

$$1 + \left(\frac{dy}{dx}\right)^2 + (y - \nu)\frac{d^2y}{dx^2} = 0 \quad (2),$$

of Art. 180. Putting in Eq. 2, for $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$, their values, it becomes

$$a^4y^3 + b^4x^2y - a^2b^4(y - \nu) = 0:$$

whence $y - \nu = \dfrac{(a^4y^2 + b^4x^2)y}{a^2b^4} = \dfrac{a^4y^2 + b^2(a^2b^2 - a^2y^2)}{a^2b^4} y$

$$= \frac{(a^2 - b^2)y^2 + b^4}{b^4} y.$$

Making $a^2 - b^2 = c^2$, we find

$$y - \nu = \frac{(b^4 + c^2y^2)y}{b^4} = y + \frac{c^2y^3}{b^4} : \therefore \nu = -\frac{c^2y^3}{b^4} \quad (3).$$

Substituting in (1) the value of $y - \nu$ just found, we get, after reduction, and the elimination of y by means of the equation of the ellipse, $\mu = \dfrac{c^2x^3}{a^4}$ (4).

Eq. 4 might have been derived from (3) by changing the sign of the latter, and in it writing x for y, and a for b. This is a consequence of the symmetry of the equation of the ellipse, and the relation between a, b, and c.

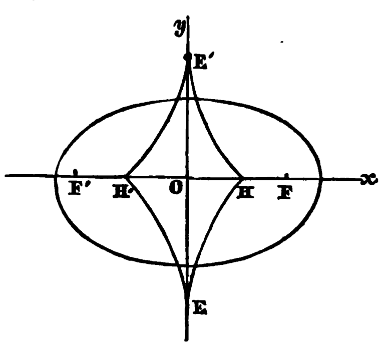
Put $\dfrac{c^2}{a} = m$, $\dfrac{c^2}{b} = n$; then $\dfrac{x}{a} = \left(\dfrac{\mu}{m}\right)^{\frac{1}{3}}$, $\dfrac{y}{b} = \left(\dfrac{\nu}{n}\right)^{\frac{1}{3}}$.

Writing the equation of the ellipse under the form

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

this becomes, when the above values of $\dfrac{x}{a}$, $\dfrac{y}{b}$, are substituted,

$$\left(\frac{\mu}{m}\right)^{\frac{2}{3}} + \left(\frac{\nu}{n}\right)^{\frac{2}{3}} = 1$$



for the equation of the evolute. The form of this equation shows that the curve is symmetrical with respect to the axes of the ellipse. For $\nu = 0$ we have

$$\mu = \pm m = \pm \frac{c^2}{a}.$$

The curve has, therefore, two points, H, H′, in the transverse

axis, situated between the foci, and equidistant from the centre. Making $\mu = 0$, we find $\nu = \pm n = \pm \frac{c^2}{b}$ for the distances from the centre to the points E, E', at which the curve meets the conjugate axis.

By two differentiations, we find

$$\frac{1}{m}\left(\frac{\mu}{m}\right)^{-\frac{1}{3}} + \frac{1}{n}\left(\frac{\nu}{n}\right)^{-\frac{1}{3}} \frac{d\nu}{d\mu} = 0,$$

$$-\frac{1}{m^2}\left(\frac{\mu}{m}\right)^{-\frac{4}{3}} - \frac{1}{n^2}\left(\frac{\nu}{n}\right)^{-\frac{4}{3}}\left(\frac{d\nu}{d\mu}\right)^2 + 3\,\frac{1}{n}\left(\frac{\nu}{n}\right)^{-\frac{1}{3}}\frac{d^2\nu}{d\mu^2} = 0:$$

whence

$$\frac{d^2\nu}{d\mu^2} = \frac{\frac{1}{m^2}\left(\frac{\mu}{m}\right)^{-\frac{4}{3}} + \frac{1}{n^2}\left(\frac{\nu}{n}\right)^{-\frac{4}{3}}\left(\frac{d\nu}{d\mu}\right)}{3\,\frac{1}{n}\left(\frac{\nu}{n}\right)^{-\frac{1}{3}}}.$$

Since the numerator of this expression is positive, the sign of $\frac{d^2\nu}{d\mu^2}$ will be the same as that of the denominator; that is, $\frac{d^2\nu}{d\mu^2}$ and ν will have the same sign. The evolute at all its points will therefore be convex towards the axis of x (Art. 155). Moreover, we have

$$\frac{d\nu}{d\mu} = -\frac{\left(\frac{\mu}{m}\right)^{-\frac{1}{3}}}{\left(\frac{\nu}{n}\right)^{-\frac{1}{3}}}\,\frac{\frac{1}{m}}{\frac{1}{n}} = -\left(\frac{m\nu}{n\mu}\right)^{\frac{1}{3}}\frac{n}{m}.$$

Since this differential co-efficient becomes zero for $\nu = 0$, and infinite for $\mu = 0$, we conclude that the axes of the ellipse are tangents to the evolute at the points H, H', and E, E'; and that, in consequence of the symmetry of the curve with respect to the axes, these points are cusps.

182. Radius of curvature, and evolute of the parabola.

When referred to the principal vertex as the origin of co-

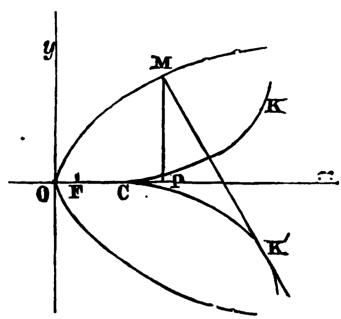
ordinates, the equation of the parabola is $y^2 = 2px$, from which we find

$$\frac{dy}{dx} = \frac{p}{y}, \quad \frac{d^2y}{dx^2} = -\frac{p^2}{y^3};$$

and, by means of these, the general value of ρ, Art. 171, becomes, without respect to sign,

$$\rho = \frac{\left(1 + \frac{p^2}{y^2}\right)^{\frac{3}{2}}}{\frac{p^2}{y^3}} = \frac{(y^2 + p^2)^{\frac{3}{2}}}{p^2}.$$

To get the equation of the evolute, we must substitute the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, in



$$x - \mu + (y - \nu)\frac{dy}{dx} = 0,$$

$$1 + \left(\frac{dy}{dx}\right)^2 + (y - \nu)\frac{d^2y}{dx^2} = 0,$$

which thus become

$$x - \mu + (y - \nu)\frac{p}{y} = 0,$$

$$1 + \frac{p^2}{y^2} - (y - \nu)\frac{p^2}{y^3} = 0.$$

The elimination of x and y between these equations and the equation of the parabola leads to the equation of the evolute. From the second, we find

$$1 + \frac{p^2}{y^2} - \frac{p^2}{y^2} + \frac{p^2\nu}{y^3} = 0; \quad \therefore \nu = -\frac{y^3}{p^2}:$$

and, putting this value of ν in the first, we have

$$x - \mu + p + \frac{y^2}{p} = 0; \quad \therefore \mu - p = 3x.$$

Therefore we have

$$y^3 = -p^2\nu,\ x = \frac{1}{3}(\mu - p),\ y^2 = 2px:$$

whence $$y^6 = p^4\nu^2,\ y^6 = (2px)^3 = \left\{\frac{2}{3}p(\mu - p)\right\}^3;$$

and $$p^4\nu^2 = \frac{8}{27}p^3(\mu - p)^3,\ \nu^2 = \frac{8}{27p}(\mu - p)^3$$

for the required equation.

If the origin of co-ordinates be transferred to a point at the distance p in the direction of positive abscissæ, the new being parallel to the primitive axes, the equation of the evolute takes the form

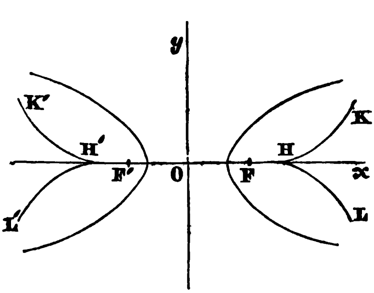
$$\nu^2 = \frac{8}{27p}\mu^3, \text{ or } \nu = \pm\sqrt{\frac{8}{27p}}\mu^{\frac{3}{2}}.$$

We readily recognize that this curve is symmetrical with respect to the axis of abscissæ, and that it extends without limit in the direction of x positive.

By differentiation, we find

$$\frac{d\nu}{d\mu} = \frac{3}{2}\sqrt{\frac{8}{27p}}\mu,\ \frac{d^2\nu}{d\mu^2} = \frac{3}{4}\sqrt{\frac{8}{27p}}\mu^{-\frac{1}{2}} = \frac{1}{\sqrt{6p}}\frac{1}{\sqrt{\mu}}.$$

Therefore, at the origin of co-ordinates, the axis of x is tangent to the curve, and this point is a cusp; and, since the sign of $\frac{d^2\nu}{d\mu^2}$ is the same as that of ν, the curve is at all points convex towards the axis of x.



183. The expression for the radius of curvature and the equation of the evolute of the hyperbola may be deduced from those for the ellipse by changing b^2 into $-b^2$. Thus we have, for the radius of curvature,

$$\rho = \frac{(b^4x^2 + a^4y^2)^{\frac{3}{2}}}{a^4b^4},$$

and, for the equation of the evolute,

$$\left(\frac{\mu}{m}\right)^{\frac{2}{3}} - \left(\frac{\nu}{n}\right)^{\frac{2}{3}} = 1,$$

after making $c^2 = a^2 + b^2$, $\frac{c^2}{a} = m$, $\frac{c^2}{b} = n$.

The form of this equation shows that the evolute of the hyperbola is composed of two branches of unlimited extent, and symmetrical with respect to both axes of the hyperbola. It has two cusps situated on the transverse axis beyond the foci, and is convex at all points towards the transverse axis.

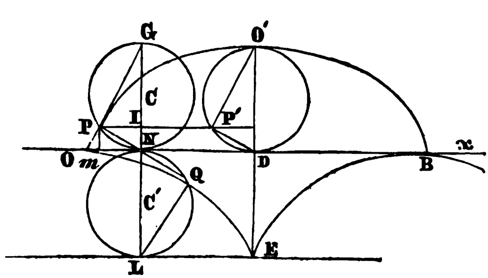
184. Radius of curvature and evolute of the cycloid.

By squaring the value of $\frac{dy}{dx}$, which, for this curve, is $\frac{dy}{dx} = \sqrt{\frac{2r - y}{y}}$ (Art. 146), and differentiating, we find

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = -\frac{2r}{y^2}\frac{dy}{dx}: \therefore \frac{d^2y}{dx^2} = -\frac{r}{y^2}.$$

Substituting these values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, in the general expression for ρ, we have

$$\rho^2 = \frac{\left(\frac{2r}{y}\right)^3}{\frac{r^2}{y^4}} = \frac{8r^3y}{r^2}: \therefore \rho = 2\sqrt{2ry}.$$



Now, $Pm = IN$, and, from the right-angled triangle PNG, we have

$PN = \sqrt{GN \times NI}$;

that is, $PN = \sqrt{2ry}$.

Hence the radius

of curvature at any point of the cycloid is twice the normal at that point; and, if PN be produced until $NQ = PN$, the point Q will be the centre of curvature.

185. The property just demonstrated leads, by very simple deductions, to the determination of the evolute of the cycloid.

Produce the vertical diameter GN of the generating circle (figure last article), making $NL = GN$, and on NL, as a diameter, describe a circle. Through L, the lower extremity of this diameter, draw LE parallel to Ox, meeting the axis $O'D$, produced in E. The arcs PN, NQ, belonging to equal chords, are equal; $\therefore$ arc $NQ = ON$: but $OD =$ arc NQL; $\therefore$ arc $LQ = ND = LE$. Thus it is seen, that if two equal circles lying in the same plane be tangent to each other, and the one be rolled on the common tangent while the other is rolled on a parallel to it at the distance of the diameter of the circle, the points of the two circumferences which are common at the time of starting will, during the motion, generate two equal cycloids; that generated by the point in the circumference of the second circle being the evolute of that generated by the point in the first.

This relation between the two cycloids, generated as just described, may also be inferred from the property of the supplementary chords of the generating circle, which are drawn through the extremities of the vertical diameter of this circle in any of its positions, and the corresponding point of the cycloid (Art. 146). For, since PG is tangent to the cycloid $OO'B$ at the point P, NQ, or PN produced, is tangent to the cycloid OQE at the point Q. Hence this last curve is the locus of the intersections of the consecutive normals to the cycloid $OO'B$, and is therefore its evolute.

186. The application of the formulas of Art. 180 leads to the same result.

From the equation of the cycloid, we have

$$\frac{dy}{dx} = \sqrt{\frac{2r - y}{y}}, \quad \frac{d^2y}{dx^2} = -\frac{r}{y^2}.$$

Substituting these values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, in the equations

$$x - \mu + (y - \nu)\frac{dy}{dx} = 0,$$

$$1 + \left(\frac{dy}{dx}\right)^2 + (y - \nu)\frac{d^2y}{dx^2} = 0,$$

we find from the second

$$\frac{2r}{y} - (y - \nu)\frac{r}{y^2} = 0, \text{ or } 2ry - r(y - \nu) = 0;$$

$$\therefore \qquad y = -\nu \quad (1);$$

and from the first

$$x - \mu + (y - \nu)\sqrt{\frac{2r - y}{y}} = 0.$$

which, if we replace y by the value just found for it, and transpose, becomes

$$x = \mu + 2\nu\sqrt{-\frac{2r + \nu}{\nu}} \quad (2).$$

The equation of the cycloid

$$x = r\cos.^{-1}\frac{r - y}{r} - \sqrt{2ry - y^2} \quad (3),$$

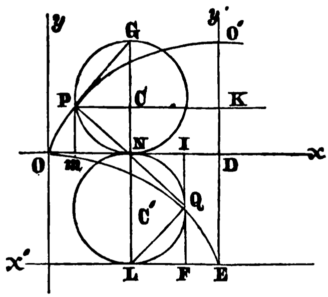
by the substitution of these values of x and y, becomes

$$\mu + 2\nu\sqrt{-\frac{2r + \nu}{\nu}} = r\cos.^{-1}\frac{r + \nu}{r} - \sqrt{-2r\nu - \nu^2} \quad (4),$$

which is the equation of the evolute. But from Eqs. 1 and 2, it is seen, as it also is from (4), that there are no points of the

curve for which ν is positive. Making ν negative, transposing and reducing, we have, finally,

$$\mu = r\cos.^{-1}\frac{r-\nu}{r} + \sqrt{2r\nu - \nu^2} \quad (5).$$



Now, let the reference of the curve be changed from Ox, Oy, to Ex', Ey'; positive abscissæ being estimated from E towards x', and positive ordinates from E towards y', x_1 and y_1 denoting the new co-ordinates of the evolute. Since $OD = \pi r$, $DE = 2r$, we have

$$\mu = OD - DI = \pi r - x_1, \quad \nu = IF - FQ = 2r - y_1.$$

Eq. 5, by the substitution of these values of μ and ν, becomes

$$\pi r - x_1 = r\cos.^{-1}\frac{r-(2r-y_1)}{r} + \sqrt{2r(2r-y_1) - (2r-y_1)^2},$$

which reduces to

$$\pi r - x_1 = r\cos.^{-1}\frac{y_1 - r}{r} + \sqrt{2ry_1 - y_1^2},$$

or

$$x_1 = r\left(\pi - \cos.^{-1}\frac{y_1 - r}{r}\right) - \sqrt{2ry_1 - y_1^2}.$$

But $\cos.^{-1}\frac{y_1 - r}{r} = \pi - \cos.^{-1}\frac{r - y_1}{r}$: introducing this in the equation above, it becomes, finally,

$$x_1 = r\cos.^{-1}\frac{r - y_1}{r} - \sqrt{2ry_1 - y_1^2}.$$

This equation differs in no respect from (3), except in having x_1, y_1, instead of x and y; which shows that the evolute of a cycloid is an equal cycloid, situated, with reference to the axes Ex', Ey', as the involute is with respect to the axes Ox, Oy.

187. It has been proved (Art. 180) that the length of an

arc of the evolute to any curve is the difference of the radii of curvature corresponding to the extreme points of the arc. In the cycloid at the point O (last figure), $\rho = 2\sqrt{2ry} = 0$: hence $PQ = 2PN$ is the length of the arc OQ; and, with respect to the given cycloid, arc $PO' = 2PG$.

To express the arc PO' in terms of the ordinate of the point P, we have

$$\text{arc } PO' = 2PG = 2\sqrt{2r \times GC}.$$

But $\quad GC = 2r - y$: $\therefore$ arc $PO' = 2\sqrt{4r^2 - 2ry}.$

Making $y = 0$, in this value of PO', we have arc $O'O = 4r$: hence *the entire arc of the cycloid is four times the diameter of the generating circle.*

188. Envelopes. If one or more of the constants entering the equation of a curve be changed in value, we shall have a new curve, differing in position and dimensions from the given curve, but agreeing with it in kind: that is, if the given curve be an ellipse, the new curve will be an ellipse; if a parabola, the new curve will be a parabola. The constants which thus change in value are called the *variable parameters* of the curve represented by the equation.

The locus of the intersections, if any, of the consecutive curves of the same species,—that is, of curves whose equations are derived from a given equation by causing one or more of its constants to vary by continuous degrees,—is called an *envelope.*

Suppose $F(x, y, a) = 0$ (1) to be the equation of a curve involving, among others, the constant a; and let a be taken as the variable parameter. Changing a into $a + h$, the equation becomes $F(x, y, a + h) = 0$ (2), which represents another curve belonging to the family of that represented by (1).

By Art. 56, Eq. 2 may be put under the form

$$F(x, y, a) + hF'(x, y, a + \theta h) = 0 \quad (3).$$

Observing that F' signifies the derivative, with respect to a, of the function symbolized by F, Eqs. 1 and 3, when simultaneous, are equivalent to

$$F(x, y, a) = 0, \;\; F'(x, y, a + \theta h) = 0 \quad (4);$$

and the values of x and y, determined by the combination of these equations, will be the co-ordinates of the intersection of the curves of which (1) and (3) are the equations.

If h be diminished without limit, Eqs. 4 become

$$F(x, y, a) = 0, \;\; F'(x, y, a) = 0 \quad (5);$$

and the point determined by these equations is the limit of the intersections of the curves of which (1) and (2) are the equations. The equation which results from the elimination of a between Eqs. 5 will evidently be the envelope of the family of curves represented by the equation $F(x, y, a) = 0$, and of which the individual curves are formed by assigning different values to a.

The envelope touches each curve of the series at the point common to the curve and the envelope. This is proved by showing that the envelope and the curve, at the common point, have the same tangent.

Since (1) becomes the equation of the envelope when in it the value of a, deduced from the second of Eqs. 5, is substituted, let (1) be differentiated under this supposition, treating x as the independent variable, and a as a function of x and y, and we have for finding the value of $\frac{dy}{dx}$ for the envelope,

$$\frac{dF}{dx} + \frac{dF}{dy}\frac{dy}{dx} + \frac{dF}{da}\left\{\frac{da}{dx} + \frac{da}{dy}\frac{dy}{dx}\right\} = 0 \quad (6).$$

But, at the point of intersection of the envelope with the given curve

$$\frac{dF}{da} = F'(x, y, a) = 0;$$

hence (6) reduces to

$$\frac{dF}{dx} + \frac{dF}{dy}\frac{dy}{dx} = 0 \quad (7),$$

which is the same as that obtained by the differentiation of (1): whence, at the common point, the tangent line to the envelope is also a tangent line to the given curve.

Ex. 1. Find the envelope of the family of straight lines derived from the equation $y = ax + \frac{m}{a}$, by causing a to vary.

Differentiating with respect to a, x and y being constant, we have

$$x - \frac{m}{a^2} = 0; \quad \therefore a = \pm \sqrt{\frac{m}{x}};$$

$$\therefore \qquad y = \pm 2\sqrt{mx}, \quad y^2 = 4mx:$$

hence the envelope is a parabola.

Ex. 2. Find the envelope of the straight lines represented by the equation $y = ax + (b^2a^2 + c^2)^{\frac{1}{2}}$, when a is made to vary.

Differentiating with respect to a, we find

$$0 = x + \frac{b^2a}{(b^2a^2 + c^2)^{\frac{1}{2}}}: \quad \therefore a = -\frac{c}{b}\frac{x}{(b^2 - x^2)^{\frac{1}{2}}}.$$

Substituting this value of a in the given equation, we have, after reduction,

$$\frac{x^2}{b^2} + \frac{y^2}{c^2} = 1,$$

which is the equation of an ellipse referred to its centre and axes.

In each of the examples just given, it has been required to determine the curve from the general equation of the tangent

line. This process, being the inverse of that for finding the equation of the tangent line, is sometimes called "the inverse method of tangents."

If a point be taken on the axis of x at a distance from the origin equal to m, and a line be drawn through this point, making, with the axis of x, an angle having $-\frac{1}{a}$ for its tangent, the equation of this line is $y = -\frac{1}{a}(x-m)$, and it intersects the axis of y at the distance $\frac{m}{a}$ from the origin. The equation of the perpendicular to this line, at its point of intersection with the axis of y, is $y = ax + \frac{m}{a}$. Hence the geometrical interpretation of Ex. 1 is, "From a point in the axis of x, at the distance m from the origin, draw lines intersecting the axis of y, and to these, at their points of intersection with the axis of y, draw perpendiculars; required the envelope of these perpendiculars:" and that of Ex. 2, "To find the envelope of a series of straight lines, so drawn that the product of the two ordinates of any one of these lines corresponding to the abscissæ, $+b$, $-b$, shall be equal to c^2."

Ex. 3. Find the envelope of all the parabolas given by the equation $y = ax - \frac{1+a^2}{2p}x^2$, by causing a to vary.

Differentiating with respect to a, we have

$$0 = x - \frac{ax^2}{p}; \therefore a = \frac{p}{x}:$$

whence, by substituting this value of a in the given equation, we find, for the envelope,

$$x^2 = 2p\left(\frac{p}{2} - y\right), \text{ or } x^2 + 2py - p = 0,$$

which is the equation of a parabola.

Ex. 4. Find the envelope of the normals drawn to the different points of a given curve.

Let the equation of the curve be $y = f(x)$; then the equation of the normal is

$$x_1 - x + (y_1 - y)\frac{dy}{dx} = 0 \quad (1),$$

in which x_1, y_1, are the running co-ordinates of the normal.

From the equation $y = f(x)$, y and $\frac{dy}{dx}$ can be expressed in terms of x, and thus x becomes the variable parameter in Eq. 1. Hence the equation of the required envelope may be found by eliminating x between (1), and

$$-1 + (y_1 - y)\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0 \quad (2),$$

which we get by differentiating (1) with respect to x.

Comparing (1) and (2) with the formulas, Art. 171, it is seen that x_1, y_1, are the co-ordinates of the centre of curvature of the point (x, y) of the given curve; that is, *the envelope of the normals of a curve is the evolute of the curve.*

189. When the equation, representing the family of curves whose envelope is sought, involves several, say n variable parameters, and these parameters are connected by $n-1$ independent equations, instead of effecting the elimination of $n-1$ parameters, and then differentiating with respect to that which remains, we may proceed as follows: Let the equation of the curve be

$$F(x, y, a, b, c \ldots) = 0 \quad (1),$$

and let the $n-1$ equations of condition for the parameter be

$$\left.\begin{array}{r} f_1(a, b, c \ldots) = 0 \\ f_2(a, b, c \ldots) = 0 \\ \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\ f_{n-1}(a, b, c \ldots) = 0 \end{array}\right\} \quad (2).$$

By reason of Eqs. 2, $n-1$ of the parameters may be regarded as functions of the remaining one taken as independent. Let this be a, and differentiate Eqs. 1 and 2 with respect to it, thus getting

$$\frac{dF}{da}+\frac{dF}{db}\frac{db}{da}+\frac{dF}{dc}\frac{dc}{da}+, \ldots=0 \quad (3),$$

$$\left.\begin{array}{l}
\dfrac{df_1}{da}+\dfrac{df_1}{db}\dfrac{db}{da}+\dfrac{df_1}{dc}\dfrac{dc}{da}+\cdots=0 \\
\dfrac{df_2}{da}+\dfrac{df_2}{db}\dfrac{db}{da}+\dfrac{df_2}{dc}\dfrac{dc}{da}+\cdots=0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\dfrac{df_{n-1}}{da}+\dfrac{df_{n-1}}{db}\dfrac{db}{da}+\dfrac{df_{n-1}}{dc}\dfrac{dc}{da}+\cdots=0
\end{array}\right\} \quad (4).$$

Now, it is plain that if, in Eqs. 1 and 3, all the variable parameters and their functions be expressed in terms of a, and a be then eliminated between these two equations, the resulting equation will be that of the envelope. To effect this elimination, we have $2n$ equations; viz., the n given equations, and their n differential equations: but there are only $2n-1$ quantities to eliminate; viz., the n quantities $a, b, c \ldots$, and the $n-1$ quantities $\frac{db}{da}, \frac{dc}{da} \ldots$: hence the elimination is possible.

Multiply the first of Eqs. 4 by the indeterminate λ_1, the second by λ_2, and so on, and add the results and Eq. 3 together: we thus get

$$\left.\begin{array}{l}
\dfrac{dF}{da}+\lambda_1\dfrac{df_1}{da}+\lambda_2\dfrac{df_2}{da}+\cdots+\lambda_{n-1}\dfrac{df_{n-1}}{da} \\
+\left(\dfrac{dF}{db}+\lambda_1\dfrac{df_1}{db}+\lambda_2\dfrac{df_2}{db}+\cdots+\lambda_{n-1}\dfrac{df_{n-1}}{db}\right)\dfrac{db}{da} \\
+\left(\dfrac{dF}{dc}+\lambda_1\dfrac{df_1}{dc}+\lambda_2\dfrac{df_2}{dc}+\cdots+\lambda_{n-1}\dfrac{df_{n-1}}{dc}\right)\dfrac{dc}{da} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\}=0 \quad (5).$$

By means of the $n-1$ indeterminate multipliers $\lambda_1, \lambda_2 \ldots, \lambda_{n-1}$, we may satisfy $n-1$ conditions. Let these be that the co-efficients of $\frac{db}{da}, \frac{dc}{da} \ldots$, in Eq. 5, shall reduce to zero. These, together with that expressed by Eq. 5 itself, lead to

$$\left.\begin{array}{l} \frac{dF}{da}+\lambda_1\frac{df_1}{da}+\lambda_2\frac{df_2}{da}+\cdots+\lambda_{n-1}\frac{df_{n-1}}{da}=0 \\ \frac{dF}{db}+\lambda_1\frac{df_1}{db}+\lambda_2\frac{df_2}{db}+\cdots+\lambda_{n-1}\frac{df_{n-1}}{da}=0 \\ \frac{dF}{dc}+\lambda_1\frac{df_1}{dc}+\lambda_2\frac{df_2}{dc}+\cdots+\lambda_{n-1}\frac{df_{n-1}}{dc}=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \quad (6).$$

We have now the $2n-1$ quantities $a, b, c \ldots, \lambda_1, \lambda_2 \ldots, \lambda_{n-1}$, to eliminate between the $2n$ equations (1), (2), and (6); and the result, being an equation between x and y only, will be the equation of the required envelope.

190. When the general equation of the family of curves contains only two variable parameters, and they are connected by one equation, the process admits of the simplification, and the result takes a form the same as those in Art. 128.

Ex. 1. Find the envelope to the different positions of a straight line of a given length extending from the axis of x to the axis of y.

Let c be the length of the line, and a and b be the intercepts on the axes of x and y respectively; then the equation of the line is

$$\frac{x}{a}+\frac{y}{b}=1 \quad (1),$$

and the equation connecting a and b is

$$a^2+b^2=c^2 \quad (2).$$

Differentiating (1) and (2) with respect to a and b, a being taken as independent, we have

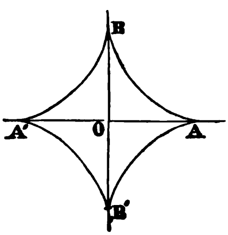
$$\frac{x}{a^2}+\frac{y}{b^2}\frac{db}{da}=0,\quad a+b\frac{db}{da}=0,$$

and therefore, according to Art. 128,

$$\frac{\frac{x}{a^2}}{a}=\frac{\frac{y}{b^2}}{b}=\frac{\frac{x}{a}}{a^2}=\frac{\frac{y}{b}}{b^2}=\frac{1}{c^2};$$

whence $a=x^{\frac{1}{3}}c^{\frac{2}{3}}$, $b=y^{\frac{1}{3}}c^{\frac{2}{3}}$, and

$$a^2+b^2=c^2=(x^{\frac{2}{3}}+y^{\frac{2}{3}})\,c^{\frac{4}{3}}:\ \therefore\ x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$$



is the equation of the envelope. The figure represents the curve traced in the several angles of the co-ordinate axes.

Ex. 2. Find the envelope of the series of ellipses formed by varying a and b in the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b}=1,$$

a and b being subject to the condition $ab=c^2$. By differentiating with respect to a and b, regarding a as independent, we have

$$\frac{x^2}{a^3}+\frac{y^2}{b^3}\frac{db}{da}=0,\quad \frac{1}{a}+\frac{1}{b}\frac{db}{da}=0;$$

$$\therefore\qquad \frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{1}{2};\ \therefore\ a=x\sqrt{2},\ b=y\sqrt{2}:$$

whence

$$xy=\frac{c^2}{2},$$

which is the equation of an hyperbola referred to its centre, and asymptotes as axes.

EXAMPLES.

1. What is the radius of curvature of the curve

$$y = x^4 - 4x^3 - 18x^2$$

at the origin of co-ordinates?

Ans. $\rho = \frac{1}{36}$.

2. Find the parabola which has the most intimate contact with the curve $y = \frac{x^3}{a^2}$ at the point having a for its abscissa, the axis of the parabola being parallel to the axis of y.

Ans. $\left(x - \frac{a}{2}\right)^2 = \frac{a}{3}\left(y - \frac{a}{4}\right)$.

3. Show that, at one of the points where $y = 0$ in the curve

$$y^2 = \frac{ax(x - 3a)}{x - 4a},$$

the radius of curvature is $\frac{3a}{8}$; and at the other, $\frac{3a}{2}$.

4. What is the radius of curvature of the spiral of Archimedes, the polar equation of this spiral being $r = a\theta$?

Ans. $\rho = \pm\frac{(a^2 + r^2)^{\frac{3}{2}}}{2a^2 + r^2}$.

5. The Lemniscata of Bernoulli is the locus of the points in which the tangents at the different points of an equilateral hyperbola are intersected by the perpendiculars let fall upon them from the centre of the hyperbola. Its polar equation is $r^2 = a^2 \cos. 2\theta$. What are the radius of curvature and the chord of curvature at any point of this curve?

Ans. $\rho = \frac{a^2}{3r}$; chord of curvature $= \frac{2}{3}r$.

6. If a curve have $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$ for its equation, prove that the general co-ordinates of its centre of curvature are

$$x_1 = x - y\sqrt{\frac{y^2}{c^2} - 1}, \quad y_1 = 2y.$$

7. What is the envelope of all ellipses having a constant area, the axes being coincident?

Ans. $4x^2y^2 = c^4$; πc^2 being the given area.

8. Find the envelope of the curves represented by the equation

$$\left(\frac{x-a}{h}\right)^2 + \left(\frac{y-b}{k}\right)^2 = 1,$$

a and b being the variable parameters connected by the equation

$$\left(\frac{a}{h}\right)^2 + \left(\frac{b}{k}\right)^2 = 1.$$

Ans. $\dfrac{x^2}{h^2} + \dfrac{y^2}{k^2} = 4.$

9. Find the envelope of the system of straight lines connecting, pair by pair, the feet of the perpendiculars let fall from the different points of an ellipse upon its axes; the equation of the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Ans. $\left(\dfrac{x}{a}\right)^{\frac{2}{3}} + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} = 1.$

10. What is the envelope of the series of circles, the circumferences of which pass through the origin, and which have their centres on the curve of which the equation is

$$a^2y^2 - b^2(2ax - x^2) = 0?$$

Ans. $(x^2 + y^2 - 2ax)^2 - 4a^2x^2 - 4b^2y^2 = 0.$

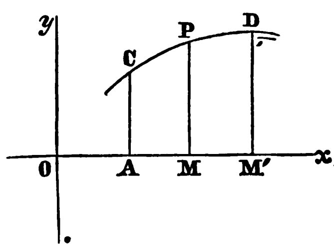
INTEGRAL CALCULUS.

SECTION I.

MEANING OF INTEGRATION. — NOTATION. — DEFINITE AND INDEFINITE INTEGRALS. — DIRECT INTEGRATION OF EXPLICIT FUNCTIONS OF A SINGLE VARIABLE. — INTEGRATION OF A SUM. — INTEGRATION BY PARTS. — BY SUBSTITUTION.

191. Any given function of a single variable may always be regarded as the differential co-efficient of some other function of the same variable; that is, there is some second function, which, when differentiated, will have the given function for its differential co-efficient.

For let $f(x)$ be the given function. If this admits of possible values for real values of x, we may construct the curve CPD, which, referred to the rectangular axes Ox, Oy, has $y = f(x)$ for its equation. The area included between this curve and the axis of x, that is



limited on the one side by the fixed ordinate CA, corresponding to $x = a$, and on the other by the ordinate PM, corresponding to the variable abscissa x, is evidently a function of

x; and, of this function, y or $f(x)$ is the differential co-efficient (Art. 164): hence we should have

$$\frac{d}{dx}(\text{area } ACPM) = f(x),$$

$$\frac{d}{dx}(\text{area } ACPM)\,dx = f(x)\,dx.$$

192. It will be found that the operations of the Integral Calculus are mainly those of passing from given functions to others, which, by differentiation, would produce the given functions. The fact that these operations are the inverse of those of the Differential Calculus has been taken as the basis of the definition of the Integral Calculus. But the fundamental proposition of the Integral Calculus is the summation of a certain infinite series of infinitely small terms. To effect this summation, we must generally know the function of which a given function is the differential co-efficient. The proposition may be stated thus: —

Let $f(x)$ be a function of x, which is finite and continuous for all values of x between x_0, x_n, and of invariable sign between these limits. Let x_n be greater than x_0, and divide the difference $x_n - x_0$ into a number n of parts, equal or unequal, represented by $x_1 - x_0$, $x_2 - x_1$, $x_3 - x_2 \ldots$, $x_n - x_{n-1}$; required the sum of the series

$$S = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \cdots$$
$$\cdots + f(x_{n-1})(x_n - x_{n-1}),$$

when the number of parts into which $x_n - x_0$ is divided is increased without limit, or n is made infinite. For brevity, denote the intervals $x_1 - x_0$, $x_2 - x_1 \ldots$, $x_n - x_{n-1}$, by h_1, $h_2 \ldots$, h_n, and the series becomes

$$S = f(x_0)h_1 + f(x_1)h_2 + \cdots + f(x_{n-2})h_{n-1} + f(x_{n-1})h_n \quad (1).$$

Now suppose $F(x)$ to be the function of x, of which $f(x)$ is the first derived function; then

$$\lim. \frac{F(x+h)-F(x)}{h} = f(x).$$

But, before passing to the limit, we should have

$$\frac{F(x+h)-F(x)}{h} = f(x) + \rho \text{ (Art. 15)},$$

ρ being a quantity that vanishes with h: therefore

$$F(x+h) - F(x) = h\left\{f(x)+\rho\right\} \quad (2).$$

In (2), giving to h the values $h_1, h_2 \ldots, h_n$, and to x the values $x_0, x_1 \ldots, x_{n-1}, x_n$, and denoting the corresponding values of ρ by $\rho_1, \rho_2 \ldots, \rho_n$, observing that

$$x_0 + h_1 = x_1, \ x_1 + h_2 = x_2 \ldots,$$

we have

$$F(x_1) - F(x_0) = h_1\left\{f(x_0) + \rho_1\right\},$$
$$F(x_2) - F(x_1) = h_2\left\{f(x_1) + \rho_2\right\},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$F(x_{n-1}) - F(x_{n-2}) = h_{n-1}\left\{f(x_{n-2}) + \rho_{n-1}\right\},$$
$$F(x_n) - F(x_{n-1}) = h_n\left\{f(x_{n-1}) + \rho_n\right\}.$$

Adding these equations member to member, for the first member of the result, we have $F(x_n) - F(x_0)$. The second member is composed of two series, the terms of one being of the form $hf(x)$; and of the other, $h\rho$. Denote the sum of the terms of the first series by $\Sigma f(x)h$, and of the second by $\Sigma\rho h$; then our result may be written

$$F(x_n) - F(x_0) = \Sigma f(x)\,h + \Sigma\rho h \quad (3).$$

If ρ', the greatest among the quantities $\rho_1, \rho_2 \ldots, \rho_n$, be substituted for ρ in the series represented by $\Sigma\rho h$, we should have

$$\Sigma\rho h < \rho'(h_1 + h_2 + \cdots + h_n) = \rho'(x_n - x_0).$$

But ρ' vanishes when h is decreased without limit: *hence*

$$F(x_n) - F(x_0)$$

is the value towards which the series $\Sigma f(x)h$ *converges when the quantities of which* h *is the type are diminished without limit;* that is,

$$\lim. \Sigma f(x)h = F(x_n) - F(x_0) \quad (4).$$

193. It may be readily proved that $\Sigma f(x)h$ has a definite value when h is indefinitely decreased, and when, therefore, the number of parts into which the interval $x_n - x_0$ is divided becomes infinite. For let A_0 be the least, and A_1 the greatest, of the values assumed by $f(x)$ for values of x between x_0, x_n: then

$$\Sigma f(x)h > A_0(h_1 + h_2 + \cdots + h_n) = A_0(x_n - x_0),$$

$$\Sigma f(x)h < A_1(h_1 + h_2 + \cdots + h_n) = A_1(x_n - x_0);$$

and since, by hypothesis, both A_0 and A_1 are finite, the same is true of $\Sigma f(x)h$. It is evident that the values of $f(x)$ intermediate to A_0, A_1, will be furnished by the expression

$$f\{x_0 + \theta(x_n - x_0)\},$$

θ being a proper fraction; and that such a value can be assigned to θ as will make

$$\Sigma f(x)h = (x_n - x_0)f\{x_0 + \theta(x_n - x_0)\}$$

a true equation.

194. Putting Eq. 3 of Art. 192 under the form

$$\Sigma f(x)h = F(x_n) - F(x_0) - \Sigma\rho h,$$

it is seen that the value of $\Sigma f(x)h$ will, in general, depend on the number and value of the parts $h_1, h_2 \ldots, h_n$, into which the interval $x_n - x_0$ is divided, but that $\lim. \Sigma f(x)h$, for which $\Sigma\rho h$ vanishes, is independent of the mode of division. When

all the parts into which $x_n - x_0$ is divided are equal, each is equal to $\frac{x_n - x_0}{n}$; and any one of the intermediate values of x, as x_r, is equal to $x_0 + \frac{r}{n}(x_n - x_0)$. In this case, the value of $\lim. \Sigma f(x)h$ is represented by $\int_{x_0}^{x_n} f(x)\,dx = F(x_n) - F(x_0)$. The symbol $\int$ signifies *sum*, and dx represents the $h = \Delta x$ of the expression $\Sigma f(x)h$. The quantity

$$\int_{x_0}^{x_n} f(x)\,dx = F(x_n) - F(x_0)$$

is called a *definite integral;* the operation by which we pass from $f(x)\,dx$ to $\int_{x_0}^{x_n} f(x)\,dx$ is called *integration;* and x_n, x_0, are the *limits* of the integral. Since $F(x_n) - F(x_0)$ is the value of this definite integral, we must first find the function $F(x)$ of x, of which $f(x)$ is the differential co-efficient. The relation between $f(x)$ and $F(x)$ is expressed by

$$f(x) = \frac{d}{dx} F(x),$$

which, by the notation of the Integral Calculus, is

$$\int f(x)\,dx = F(x).$$

195. The function $F(x)$ of x, which, differentiated, would reproduce $f(x)\,dx$, is denominated *indefinite integral.* But a constant connected with a function by the sign plus or minus disappears in differentiation; therefore the more general relation between $f(x)$ and $F(x)$ is

$$\int f(x)\,dx = F(x) \pm C:$$

so that the proper value of y to verify the equation

$$dy = f(x)\,dx$$

is given by the equation

$$y = \int f(x)\,dx \pm C;$$

and the two symbols d and $\int$, the one indicating differentiation, and the other integration, neutralize each other, and we shall always have

$$\int du = u \pm C, \quad d\int du = du.$$

The constant thus added to an indefinite integral is called the *arbitrary constant of integration*, or, simply, the *arbitrary constant;* it being any quantity which does not depend on the independent variable x.

The operation of passing from an indefinite integral to a definite integral consists in substituting in the indefinite, successively, the limiting values of the independent variable, and taking the difference of the results. The arbitrary constant will, of course, disappear in the subtraction.

196. In differentiation, constant factors may be written before the sign of differentiation. The same may be done in integration. For

$$\int dau = au, \qquad a\int du = au;$$

$$\therefore \qquad \int dau = a\int du, \text{ or } \int adu = a\int du;$$

or, more generally,

$$\int af(x)\,dx = a\int f(x)\,dx.$$

Observing that $\int_{x_0}^{x_n} f(x)\,dx$ is the expression for the limit of the sum $\Sigma f(x)\Delta x$, that is, the expression for this sum, when the number of parts of the interval $x_n - x_0$ is increased without limit, and the value of the parts severally correspondingly decreased, it is evident, that, at the limit, the addition or omission of a finite number of the components $f(x)\Delta x$ of $\Sigma f(x)\Delta x$ would not affect the result.

A single term $f(x)\Delta x$ of the expression $\Sigma f(x)\Delta x$ is called an *element.*

197. Direct integration of simple functions.

We shall, for the present, confine ourselves to the determination of indefinite integrals, to which it must be understood that an arbitrary constant is to be added.

There are many cases in which a function is at once recognized to be the differential co-efficient of another. In such cases, we have simply to write the second as the integral of the first.

Subjoined is a table of the integrals of the simple functions.

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \qquad \int a^x dx = \frac{a^x}{la},$$

$$\int \sin. x dx = -\cos. x, \qquad \int e^x dx = e^x,$$

$$\int \cos. x dx = \sin. x, \qquad \int \frac{dx}{x} = lx,$$

$$\int \frac{dx}{\cos.^2 x} = \tan. x, \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin.^{-1}\frac{x}{a} = -\cos.^{-1}\frac{x}{a},$$

$$\int \frac{dx}{\sin.^2 x} = -\cot. x, \quad \int \frac{dx}{\sqrt{1 - x^2}} = \sin.^{-1} x = -\cos.^{-1} x,$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan.^{-1}\frac{x}{a} = -\frac{1}{a}\cot.^{-1}\frac{x}{a}.$$

In all of these formulas, x may be the independent variable, or it may be any function of the independent variable; for if, in the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$, x be replaced by $f(x)$, we should have

$$\int \{f(x)\}^n df(x) = \frac{\{f(x)\}^{n+1}}{n+1}.$$

When $n = -1$, the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$ reduces to

$$\int \frac{dx}{x} = \frac{1}{0} = \infty;$$

whereas, we know that $\int \frac{dx}{x} = lx$. The failure of the formula to give the true result in this case arises from the fact that the transcendental quantity lx cannot be represented by an algebraic expression. It may, however, by a suitable transformation, be made to give the true value of $\int x^n dx$ when $n = -1$. Take the general formula $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, which may be written

$$\int x^n dx = \frac{x^{n+1}}{n+1} - \frac{1}{n+1} + \frac{1}{n+1} + C.$$

Now, the term $+\frac{1}{n+1}$, in the second member, may be included in the arbitrary constant C: and thus we have

$$\int x^n dx = \frac{x^{n+1}}{n+1} - \frac{1}{n+1} + C = \frac{x^{n+1} - 1}{n+1} + C;$$

or, omitting the constant,

$$\int x^n dx = \frac{x^{n+1} - 1}{n+1} = \frac{0}{0} \text{ when } n = -1.$$

The true value of this is found by differentiating the numerator and denominator with respect to n, and taking the ratio of the differential co-efficients (Art. 101). We find

$$\left(\frac{x^{n+1} - 1}{n+1}\right)_{n=-1} = \left(\frac{x^{n+1}\, lx}{1}\right)_{n=-1} = lx.$$

198. The rules of the Differential Calculus enable us to find the differential co-efficients of all known functions; but the inverse operation, of deducing the function of which a given function is the differential co-efficient, is not always possible. Whatever the assumed function may be, there must be some other function of the quantities involved, which, differentiated, would produce it (Art. 191). The second of the two functions thus related as differential to integral may not be-

long to any of the small class of simple functions which have been admitted into analysis, or to any combination of such functions; in which case, we are limited to series and approximations for the expression of integrals. For example, we recognize $\frac{1}{\sqrt{a^2-x^2}}$ to be the differential co-efficient of $\sin.^{-1}\frac{x}{a}$, or that $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin.^{-1}\frac{x}{a}$, because the latter function has been named, and its properties investigated. Had this not been done, the integral $\int\frac{dx}{\sqrt{a^2-x^2}}$ could not have been expressed by means of a simple function.

199. Integration of a sum of functions of the same variable.

In the Differential Calculus (Art. 19), it is proved that if

$$y=f(x)\pm\varphi(x)\pm\psi(x)\pm\cdots,$$

then
$$\frac{dy}{dx}=f'(x)\pm\varphi'(x)\pm\psi'(x)\pm\cdots,$$

or
$$dy=f'(x)\,dx\pm\varphi'(x)\,dx\pm\psi'(x)\,dx\ldots:$$

whence

$$\int dy=y=\int f'(x)\,dx\pm\int\varphi'(x)\,dx\pm\int\psi'(x)\,dx\ldots.$$

Hence the integral of the sum of any number of functions is the sum of the integrals of the component functions. For example,

$$\int(Ax^m+Bx^n+Cx^p+\cdots)\,dx=\frac{Ax^{m+1}}{m+1}+\frac{Bx^{n+1}}{n+1}+\frac{Cx^{p+1}}{p+1}+\cdots;$$

also

$$\int(5x^4-7x^3+4x-3)\,dx=x^5-\frac{7}{4}x^4+2x^2-3x,$$

and
$$\int\left(\frac{x^3-4x^2+5}{x}\right)dx=\frac{1}{3}x^3-2x^2+5lx.$$

200. Integration by parts.

If u and v are functions of the same variable, we have, by differentiation,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The integration of both members of this gives

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx:$$

therefore

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx,$$

or

$$\int udv = uv - \int vdu.$$

This method of integration, by which the determination of an integral $\int udv$ is reduced to that of another $\int vdu$, is frequently employed, and is called *integration by parts.*

Ex. 1. $\int x^2 \cos. xdx.$

Put $x^2 = u$, $\cos. xdx = dv = d \sin. x$; then, by the formula,

$$\int x^2 \cos. xdx = \int x^2 d \sin. x = x^2 \sin. x - 2\int x \sin. xdx,$$

$$\int x \sin. xdx = -\int xd \cos. x = -x \cos. x + \int \cos. xdx$$
$$= -x \cos. x + \sin. x.$$

We shall therefore have, by the substitution of this value in the first integral,

$$\int x^2 \cos. xdx = x^2 \sin. x + 2x \cos. x - 2 \sin. x.$$

Ex. 2. $\int lxdx.$

Make $lx = u$, $dx = dv$; then $\int lxdx = xlx - x.$

Ex. 3. $\int x^n e^x dx.$

Making $x^n = u$, $e^x dx = dv$, the formula gives

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx,$$

and the integration of $x^n e^x dx$ is thus brought to that of $x^{n-1}e^x dx$. By another application of the formula, the expression to be integrated would become $x^{n-2}e^x dx$; so that, if n be a positive whole number, the proposition would be reduced, after n applications of the formula, to finding the integral $e^x dx = de^x$. Hence, by a series of substitutions, we should have the required integral.

Making $\quad n = 1, \ \int xe^x dx = e^x(x - 1),$

" $\quad n = 2, \ \int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$

$$= e^x(x^2 - 2x + 2).$$

201. Integration by substitution.

It is sometimes the case that a differential expression, $f(x)dx$, which is not immediately integrable, becomes so by replacing the independent variable by some function of a new variable. The function selected must be such that it shall be capable of assuming all the values of the variable for which it is substituted within the assigned limits of the integral.

Let t be the new variable, and suppose $x = \varphi(t)$; then, by the Differential Calculus,

$$\frac{dx}{dt} = \varphi'(t), \text{ or } dx = \varphi'(t)dt,$$

and
$$f(x)dx = f\{\varphi(t)\}\varphi'(t)dt:$$

whence, by integration,

$$\int f(x)dx = \int f\{\varphi(t)\}\varphi'(t)dt;$$

in which it must be remembered, that, if the first integral is to be taken between the limits a and b, the second is to be taken between the corresponding limits a' and b'.

Ex. 1. $\quad \int(ax + b)^n dx.$

Put $ax+b=t$, whence $dx=\frac{1}{a}dt$; and therefore

$$\int(ax+b)^n dx=\frac{1}{a}\int t^n dt=\frac{1}{a}\,\frac{t^{n+1}}{n+1}.$$

Replacing t by its value, we have, for the required integral,

$$\int(ax+b)^n dx=\frac{1}{a}\,\frac{(ax+b)^{n+1}}{n+1}.$$

Ex. 2. $$\int\frac{3x^2dx}{8x^3+5}.$$

Make $8x^3+5=t$, then $3x^2dx=\frac{1}{8}dt$,

and $$\int\frac{3x^2dx}{8x^3+5}=\int\frac{1}{8}\,\frac{dt}{t}=\frac{1}{8}lt:$$

therefore $$\int\frac{3x^2dx}{8x^3+5}=\frac{1}{8}l(8x^3+5).$$

It is evident from these two examples that success in effecting integration by substitution must depend on the ingenuity of the student, and his knowledge of the forms of the differentials of the simple functions.

Miscellaneous Examples.

1. $\int\frac{xdx}{\sqrt{a^2+x^2}}$. Make $\sqrt{a^2+x^2}=t$: $\therefore$ $a^2+x^2=t^2$.

$xdx=tdt$; and therefore

$$\int\frac{xdx}{\sqrt{a^2+x^2}}=\int dt=t=\sqrt{x^2+a^2}.$$

2. $$\int\sqrt{a^2-x^2}\,dx.$$

Putting $\sqrt{a^2-x^2}=u$, $x=v$, and integrating by parts (Art. 200), we have

$$\int\sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}+\int\frac{x^2dx}{\sqrt{a^2-x^2}}\quad(1).$$

But $$\int\sqrt{a^2-x^2}\,dx=\int\frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$=\int\frac{a^2\,dx}{\sqrt{a^2-x^2}}-\int\frac{x^2\,dx}{\sqrt{a^2-x^2}}\cdot \quad (2).$$

Therefore, by the addition of (1) and (2),

$$2\int\sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}+a^2\int\frac{dx}{\sqrt{a^2-x^2}};$$

and since

$$a^2\int\frac{dx}{\sqrt{a^2-x^2}}=a^2\sin.^{-1}\frac{x}{a}, \text{ Art. 197,}$$

we have finally

$$\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin.^{-1}\frac{x}{a}.$$

3. $\int\frac{dx}{\sqrt{x^2+a^2}}\cdot$ Make $\sqrt{x^2+a^2}=t-x$: $\therefore\ a^2=t^2-2tx.$

Hence, by differentiation, $dx=\frac{t-x}{t}\,dt$: therefore

$$\int\frac{dx}{\sqrt{x^2+a^2}}=\int\frac{1}{t-x}\,\frac{t-x}{t}\,dt=\int\frac{dt}{t}=lt$$

$$=l(x+\sqrt{x^2+a^2}).$$

4. $\int\frac{dx}{\sqrt{x^2-a^2}}\cdot$ Making $\sqrt{x^2-a^2}=t-x$, and proceeding as in Ex. 3, we should find

$$\int\frac{dx}{\sqrt{x^2-a^2}}=l(x+\sqrt{x^2-a^2}).$$

5. $\int\sqrt{x^2+a^2}\,dx$. Integrating by parts, Art. 200, we have

$$\int\sqrt{x^2+a^2}\,dx=x\sqrt{x^2+a^2}-\int\frac{x^2\,dx}{\sqrt{x^2+a^2}} \quad (1).$$

But $$\int\sqrt{x^2+a^2}\,dx=\int\frac{x^2+a^2}{\sqrt{x^2+a^2}}dx$$

$$=\int\frac{x^2dx}{\sqrt{x^2+a^2}}+a^2\int\frac{dx}{\sqrt{x^2+a^2}} \quad (2).$$

Therefore, by the addition of (1) and (2),

$$2\int\sqrt{x^2+a^2}\,dx=x\sqrt{x^2+a^2}+a^2\int\frac{dx}{\sqrt{x^2+a^2}}.$$

By Ex. 3,

$$a^2\int\frac{dx}{\sqrt{x^2+a^2}}=a^2l(x+\sqrt{x^2+a^2});$$

and hence

$$\int\sqrt{x^2+a^2}\,dx=\frac{1}{2}x\sqrt{x^2+a^2}+\frac{a^2}{2}l(x+\sqrt{x^2+a^2}).$$

6. $$\int\sqrt{x^2-a^2}\,dx=\frac{1}{2}x\sqrt{x^2-a^2}-\frac{a^2}{2}l(x+\sqrt{x^2-a^2}).$$

7. $\displaystyle\int\frac{dx}{\sqrt{x^2+px+q}}$. The quantity under the radical sign in the denominator may be put under the form

$$\left(x+\frac{p}{2}-\sqrt{-q+\frac{p^2}{4}}\right)\left(x+\frac{p}{2}+\sqrt{-q+\frac{p^2}{4}}\right)$$
$$=\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}:$$

hence

$$\int\frac{dx}{\sqrt{x^2+px+q}}=\int\frac{dx}{\sqrt{\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}}}.$$

Making $x+\frac{p}{2}=t$, and $q-\frac{p^2}{4}=a^2$, we have $dx=dt$, and

$$\int\frac{dt}{\sqrt{t^2+a^2}}=l(t+\sqrt{t^2+a^2}) \text{ by Ex. 3.}$$

In this last, substituting for t and a their values, we have

$$\int \frac{dx}{\sqrt{x^2+px+q}} = l\left(x+\frac{p}{2}+\sqrt{\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}}\right)$$
$$= l\left(x+\frac{p}{2}+\sqrt{x^2+px+q}\right).$$

8. $$\int \sqrt{x^2+px+q}\, dx = \int \left\{\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}\right\}^{\frac{1}{2}} dx.$$

Put $x+\frac{p}{2}=t$, $q-\frac{p^2}{4}=a^2$, then

$$\int \sqrt{x^2+px+q}\, dx = \int \sqrt{t^2+a^2}\, dt$$
$$= \frac{1}{2} t\sqrt{t^2+a^2} + \frac{a^2}{2} l(t+\sqrt{t^2+a^2}),$$

by Ex. 5. Replacing t and a^2 by their values, we have, finally,

$$\int \sqrt{x^2+px+q}\, dx = \frac{1}{2}\left(x+\frac{p}{2}\right)\sqrt{x^2+px+q}$$
$$+\frac{1}{2}\left(q-\frac{p^2}{4}\right) l\left(x+\frac{p}{2}+\sqrt{x^2+px+q}\right).$$

9. $\int \frac{dx}{\sqrt{2ax-x^2}}$. Let $x=a-t$; then $2ax-x^2=a^2-t^2$, and $dx=-dt$: therefore

$$\int \frac{dx}{\sqrt{2ax-x^2}} = -\int \frac{dt}{\sqrt{a^2-t^2}} = \cos.^{-1}\frac{t}{a}$$
$$= \cos.^{-1}\frac{a-x}{a} = \text{vers.}^{-1}\frac{x}{a}.$$

10. $\int \frac{dx}{x\sqrt{2ax-a^2}}$. Put $x=\frac{a}{1-t}$; then

$$2ax-a^2=\frac{a^2(1+t)}{1-t},\quad x\sqrt{2ax-a^2})=\frac{a}{1-t}\sqrt{\frac{a^2(1+t)}{1-t}},$$
$$dx=\frac{adt}{(1-t)^2}, \text{ and therefore}$$

$$\int \frac{dx}{x\sqrt{2ax-a^2}} = \int \frac{\frac{adt}{(1-t)^2}}{\frac{a}{1-t}\sqrt{\frac{a^2(1+t)}{1-t}}} = \frac{1}{a}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{a} \sin.^{-1} t = \frac{1}{a} \sin.^{-1} \frac{x-a}{x}.$$

11. $\int x \cos. ax dx.$ Assume $u = \frac{x}{a}$, $v = \sin. ax$; whence $dv = a \cos. ax dx$ and $\int x \cos. ax dx = \int u dv$: therefore

$$\int x \cos. ax dx = \frac{x \sin. ax}{a} - \int \frac{\sin. ax}{a} dx$$

$$= \frac{x \sin. ax}{a} + \frac{\cos. ax}{a^2}.$$

12. $\int e^{cx} \sin. ax dx.$ Put $u = \frac{\sin. ax}{c}$, $v = e^{cx}$:

$$\therefore \quad \int e^{cx} \sin. ax dx = \frac{\sin. ax}{c} e^{cx} - \int \frac{a e^{cx} \cos. ax}{c} dx.$$

But we have, in like manner,

$$\int \frac{a e^{cx} \cos. ax}{c} dx = \frac{a \cos. ax}{c^2} e^{cx} + \int \frac{a^2 \sin. ax}{c^2} e^{cx} dx:$$

hence

$$\int e^{cx} \sin. ax dx = \frac{\sin. ax}{c} e^{cx} - \frac{a \cos. ax}{c^2} e^{cx} - \int \frac{a^2 \sin. ax}{c^2} e^{cx} dx,$$

which, by transposition and reduction, becomes

$$\int e^{cx} \sin. ax dx = \frac{e^{cx}(c \sin. ax - a \cos. ax)}{a^2 + c^2}.$$

13. $$\int e^{cx} \cos. ax dx.$$

Proceeding with this as with the last example, we should find

$$\int e^{cx} \cos. ax dx = \frac{e^{cx}(c \cos. ax + a \sin. ax)}{a^2 + c^2}.$$

14. $$\int \frac{dx}{\sqrt{(q + px - x^2)}} = \int \frac{dx}{\sqrt{\left\{q + \frac{p^2}{4} - \left(x - \frac{p}{2}\right)^2\right\}}}.$$

Put $q + \frac{p^2}{4} = a^2$, $x - \frac{p}{2} = t$: $\therefore$ $dx = dt$.

Therefore

$$\int \frac{dx}{\sqrt{(q+px-x^2)}} = \int \frac{dt}{\sqrt{(a^2-t^2)}} = \sin.^{-1}\frac{t}{a}$$

$$= \sin.^{-1}\frac{x-\frac{p}{2}}{\sqrt{q+\frac{p^2}{4}}} = \sin.^{-1}\frac{2x-p}{\sqrt{4q+p^2}}.$$

15. $\int \sqrt{(q+px-x^2)}dx = \int \sqrt{\left\{q+\frac{p^2}{4}-\left(x-\frac{p}{2}\right)^2\right\}}dx.$

Making $q+\frac{p^2}{4}=a^2$, $x-\frac{p}{2}=t$, we have

$$\int \sqrt{(q+px-x^2)}dx = \int \sqrt{a^2-t^2}\,dt$$

$$= \frac{1}{2}t\sqrt{a^2-t^2}+\frac{a^2}{2}\sin.^{-1}\frac{t}{a}, \text{ by Ex. 2,}$$

$$= \frac{1}{8}(2x-p)\sqrt{4q+p^2}+\frac{1}{8}(4q+p^2)\sin.^{-1}\frac{2x-p}{\sqrt{4q+p^2}}.$$

16. $\int \frac{dx}{x\sqrt{x^2-a^2}}$. Put $x=\frac{1}{t}$; whence $dx=-\frac{dt}{t^2}$, and

$$x\sqrt{x^2-a^2} = \frac{1}{t}\sqrt{\frac{1}{t^2}-a^2} = \frac{1}{t^2}\sqrt{1-t^2a^2};$$

$$\therefore \quad \int \frac{dx}{x\sqrt{x^2-a^2}} = -\int \frac{dt}{\sqrt{1-a^2t^2}} = -\frac{1}{a}\int \frac{dt}{\sqrt{\frac{1}{a^2}-t^2}}$$

$$= -\frac{1}{a}\sin.^{-1}at = -\frac{1}{a}\sin.^{-1}\frac{a}{x}.$$

But, since

$$\sin^{-1}\frac{a}{x}+\cos.^{-1}\frac{a}{x}=\frac{\pi}{2}, \quad -\frac{1}{a}\sin.^{-1}\frac{x}{a}=\frac{1}{a}\cos.^{-1}\frac{a}{x}-\frac{1}{a}\frac{\pi}{2},$$

hence, throwing $-\frac{1}{a}\frac{\pi}{2}$ into the constant of integration, we may write

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cos.^{-1}\frac{a}{x}.$$

$$=\frac{1}{a}\sin.^{-1}t=\frac{1}{a}\sin.^{-1}\frac{x-a}{x}.$$

11. $\int x\cos.axdx$. Assume $u=\frac{x}{a}$, $v=\sin.ax$; whence $dv=a\cos.axdx$ and $\int x\cos.axdx=\int udv$: therefore

$$\int x\cos.axdx=\frac{x\sin.ax}{a}-\int\frac{\sin.ax}{a}dx$$
$$=\frac{x\sin.ax}{a}+\frac{\cos.ax}{a^2}.$$

12. $\int e^{cx}\sin.axdx$. Put $u=\frac{\sin.ax}{c}$, $v=e^{cx}$:

$$\therefore\quad \int e^{cx}\sin.axdx=\frac{\sin.ax}{c}e^{cx}-\int\frac{ae^{cx}\cos.ax}{c}dx.$$

But we have, in like manner,

$$\int\frac{ae^{cx}\cos.ax}{c}dx=\frac{a\cos.ax}{c^2}e^{cx}+\int\frac{a^2\sin.ax}{c^2}e^{cx}dx:$$

hence

$$\int e^{cx}\sin.axdx=\frac{\sin.ax}{c}e^{cx}-\frac{a\cos.ax}{c^2}e^{cx}-\int\frac{a^2\sin.ax}{c^2}e^{cx}dx,$$

which, by transposition and reduction, becomes

$$\int e^{cx}\sin.axdx=\frac{e^{cx}(c\sin.ax-a\cos.ax)}{a^2+c^2}.$$

13. $$\int e^{cx}\cos.axdx.$$

Proceeding with this as with the last example, we should find

$$\int e^{cx}\cos.axdx=\frac{e^{cx}(c\cos.ax+a\sin.ax)}{a^2+c^2}.$$

14. $$\int\frac{dx}{\sqrt{(q+px-x^2)}}=\int\frac{dx}{\sqrt{\left\{q+\frac{p^2}{4}-\left(x-\frac{p}{2}\right)^2\right\}}}.$$

Put $q+\frac{p^2}{4}=a^2$, $x-\frac{p}{2}=t$: $\therefore\ dx=dt$.

Therefore

$$\int \frac{dx}{\sqrt{(q+px-x^2)}} = \int \frac{dt}{\sqrt{(a^2-t^2)}} = \sin.^{-1}\frac{t}{a}$$

$$= \sin.^{-1}\frac{x-\frac{p}{2}}{\sqrt{q+\frac{p^2}{4}}} = \sin.^{-1}\frac{2x-p}{\sqrt{4q+p^2}}.$$

15. $\int \sqrt{(q+px-x^2)}dx = \int \sqrt{\left\{q+\frac{p^2}{4}-\left(x-\frac{p}{2}\right)^2\right\}}dx.$

Making $q+\frac{p^2}{4}=a^2$, $x-\frac{p}{2}=t$, we have

$$\int \sqrt{(q+px-x^2)}dx = \int \sqrt{a^2-t^2}\,dt$$

$$= \frac{1}{2}t\sqrt{a^2-t^2}+\frac{a^2}{2}\sin.^{-1}\frac{t}{a}, \text{ by Ex. 2,}$$

$$= \frac{1}{8}(2x-p)\sqrt{4q+p^2}+\frac{1}{8}(4q+p^2)\sin.^{-1}\frac{2x-p}{\sqrt{4q+p^2}}.$$

16. $\int \frac{dx}{x\sqrt{x^2-a^2}}$. Put $x=\frac{1}{t}$; whence $dx=-\frac{dt}{t^2}$, and

$$x\sqrt{x^2-a^2} = \frac{1}{t}\sqrt{\frac{1}{t^2}-a^2} = \frac{1}{t^2}\sqrt{1-t^2a^2}:$$

$$\therefore \quad \int \frac{dx}{x\sqrt{x^2-a^2}} = -\int \frac{dt}{\sqrt{1-a^2t^2}} = -\frac{1}{a}\frac{dt}{\sqrt{\frac{1}{a^2}-t^2}}$$

$$= -\frac{1}{a}\sin.^{-1}at = -\frac{1}{a}\sin.^{-1}\frac{a}{x}.$$

But, since

$$\sin^{-1}\frac{a}{x}+\cos.^{-1}\frac{a}{x}=\frac{\pi}{2}, \quad -\frac{1}{a}\sin.^{-1}\frac{x}{a}=\frac{1}{a}\cos.^{-1}\frac{a}{x}-\frac{1}{a}\frac{\pi}{2}:$$

hence, throwing $-\frac{1}{a}\frac{\pi}{2}$ into the constant of integration, we may write

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cos.^{-1}\frac{a}{x}.$$

$$= \frac{1}{2} l \frac{1 + \sin. x}{1 - \sin. x} = l \frac{\cos. \frac{1}{2} x + \sin. \frac{1}{2} x}{\cos. \frac{1}{2} x - \sin. \frac{1}{2} x}$$

$$= l \tan. \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

23. $$\int \frac{dx}{\sin. x \cos. x} = \int \frac{\sin.^2 x + \cos.^2 x}{\sin. x \cos. x} dx$$

$$= \int (\tan. x + \cot. x) dx$$

$$= - l \cos. x + l \sin. x = l \frac{\sin. x}{\cos. x} = l \tan. x.$$

24. $$\int \frac{dx}{\sin.^2 x \cos.^2 x} = \int \frac{\sin.^2 x + \cos.^2 x}{\sin.^2 x \cos.^2 x} dx$$

$$= \int (\sec.^2 x + \operatorname{cosec.}^2 x) dx$$

$$= \tan. x - \cot. x.$$

25. $$\int \frac{dx}{a + b \cos. x}$$

$$= \int \frac{dx}{a \left(\sin.^2 \frac{x}{2} + \cos.^2 \frac{x}{2} \right) + b \left(\cos.^2 \frac{x}{2} - \sin.^2 \frac{x}{2} \right)}$$

$$= \int \frac{\sec.^2 \frac{x}{2} dx}{a + b + (a - b) \tan.^2 \frac{x}{2}},$$

by observing that

$$\sin.^2 \frac{x}{2} + \cos.^2 \frac{x}{2} = 1, \quad \cos. x = \cos.^2 \frac{x}{2} - \sin.^2 \frac{x}{2},$$

and dividing the numerator and denominator of the result by $\cos.^2 \frac{x}{2}$. When $a > b$, the last integral may be put under the form

$$\frac{2}{a-b}\int\frac{d\tan.\frac{x}{2}}{\frac{a+b}{a-b}+\tan.^2\frac{x}{2}}=\frac{2}{a-b}\,\frac{1}{\sqrt{\frac{a+b}{a-b}}}\tan.^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan.\frac{x}{2}\right)$$

$$=\frac{2}{\sqrt{a^2-b^2}}\tan.^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan.\frac{x}{2}\right):$$

$$\therefore\quad\int\left(\frac{dx}{a+b\cos.x}\right)_{a>b}=\frac{2}{\sqrt{a^2-b^2}}\tan.^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan.\frac{x}{2}\right).$$

When $a<b$, we have

$$\int\frac{dx}{a+b\cos.x}=\frac{2}{b-a}\int\frac{d\tan.\frac{x}{2}}{\frac{b+a}{b-a}-\tan.^2\frac{x}{2}}$$

$$=\frac{1}{\sqrt{b^2-a^2}}\,l\,\frac{\sqrt{b-a}\tan.\frac{x}{2}+\sqrt{b+a}}{\sqrt{b-a}\tan.\frac{x}{2}-\sqrt{b+a}},$$

by Ex. 18.

26. $$\int\frac{dx}{a+b\sin.x}=\int\frac{dx}{a+2b\sin.\frac{x}{2}\cos.\frac{x}{2}}$$

$$=\int\frac{dx}{a\left(\sin.^2\frac{x}{2}+\cos.^2\frac{x}{2}\right)+2b\sin.\frac{x}{2}\cos.\frac{x}{2}}$$

$$=\int\frac{\sec.^2\frac{x}{2}dx}{a\left(1+\tan.^2\frac{x}{2}\right)+2b\tan.\frac{x}{2}}$$

$$=\frac{2}{a}\int\frac{d\left(\tan.\frac{x}{2}+\frac{b}{a}\right)}{\frac{a^2-b^2}{a^2}+\left(\tan.\frac{x}{2}+\frac{b}{a}\right)^2}$$

$$=\frac{2}{\sqrt{a^2-b^2}}\tan.^{-1}\left\{\frac{a}{\sqrt{a^2-b^2}}\left(\tan.\frac{x}{2}+\frac{b}{a}\right)\right\},$$

when $a > b$; but, if $a < b$, then

$$\int\frac{dx}{a+b\sin}=\frac{2}{a}\int\frac{d\left(\tan.\frac{x}{2}+\frac{b}{a}\right)}{\left(\tan.\frac{x}{2}+\frac{b}{a}\right)^2-\frac{b^2-a^2}{a^2}}$$

$$=\frac{1}{\sqrt{b^2-a^2}}\,l\,\frac{\tan.\frac{x}{2}+\frac{b}{a}-\frac{\sqrt{b^2-a^2}}{a}}{\tan.\frac{x}{2}+\frac{b}{a}+\frac{\sqrt{b^2-a^2}}{a}}$$

$$=\frac{1}{\sqrt{b^2-a^2}}\,l\,\frac{a\tan.\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan.\frac{x}{2}+b+\sqrt{b^2-a^2}}.$$

202. Rationalization and integration of irrational functions. Examples of integration by substitution have already been given: we now proceed to show under what conditions certain irrational differential expressions may, by proper substitutions, be rendered rational, and integrated by the methods previously investigated.

Let us assume the form $x^m(a+bx^n)^{\frac{p}{q}}dx$, in which m, n, p, and q are entire or fractional, positive or negative.

Put $$a+bx^n=z^q;\ \therefore\ x=\left(\frac{z^q-a}{b}\right)^{\frac{1}{n}}$$

$$dx=\frac{qz^{q-1}}{nb^{\frac{1}{n}}}(z^q-a)^{\frac{1-n}{n}}dz:$$

whence

$$\int x^m(a+bx^n)^{\frac{p}{q}}dx = \frac{q}{nb^{\frac{m+1}{n}}}\int z^{p+q-1}(z^q-a)^{\frac{m+1}{n}-1}dz.$$

Now, if $\frac{m+1}{n}$ is an integer, the binomial $(z^q-a)^{\frac{m+1}{n}-1}$ is rational in form, and may be expanded by the Binomial Formula into a finite number of terms when the exponent $\frac{m+1}{n}-1$ is positive. Each term of the expansion, being multiplied by $z^{p+q-1}dz$, will give rise to a series of monomial differentials which can be immediately integrated.

We may also write

$$\int x^m(a+bx^n)^{\frac{p}{q}}dx = \int x^{m+\frac{np}{q}}(b+ax^{-n})^{\frac{p}{q}}dx;$$

and, by comparing this with the first case, we conclude that the substitution of z^q for $b+ax^{-n}$ will reduce

$$\int x^{m+\frac{np}{q}}(b+ax^{-n})^{\frac{p}{q}}dx$$

to an expression that is immediately integrable when

$$-\frac{m+1}{n}-\frac{p}{q}$$

is a positive integer; i.e., when $\frac{m+1}{n}+\frac{p}{q}$ is a negative integer.

Hence $\int x^m(a+bx^n)^{\frac{p}{q}}dx$ may be rationalized and integrated when $\frac{m+1}{n}$ is a positive integer by substituting z^q for $a+bx^n$; and, when $\frac{m+1}{n}+\frac{p}{q}$ is a negative integer, by substituting z^q for $b+ax^{-n}$.

It will be shown in a subsequent section (2) that the integrals may also be found when $\frac{m+1}{n}$ is a negative integer in

the first case, and when $\frac{m+1}{n}+\frac{p}{q}$ is a positive integer in the second.

Ex. 1. $\int x(a+bx)^{\frac{3}{2}}$. Here $m=1$, $n=1$, $\frac{p}{q}=\frac{3}{2}$, and $\frac{m+1}{n}=2$, a positive integer.

Put $$a+bx=z^2: \therefore x=\frac{z^2-a}{b},\ dx=\frac{2zdz}{b}:$$

$$\therefore \quad \int x(a+bx)^{\frac{3}{2}}dx=\frac{2}{b^2}\int(z^2-a)z^4dz$$

$$=\frac{2}{b^2}\int(z^6-az^4)dz$$

$$=\frac{2}{b^2}\left(\frac{z^7}{7}-\frac{az^5}{5}\right)=\frac{2(a+bx)^{\frac{5}{2}}}{b^2}\left(\frac{a+bx}{7}-\frac{a}{5}\right).$$

Ex. 2. $\int\frac{x^3dx}{(a^2+x^2)^{\frac{1}{2}}}$. In this example,

$$m=3,\ n=2,\ \frac{p}{q}=-\frac{1}{2},\ \text{and}\ \frac{m+1}{n}=2.$$

Put $a^2+x^2=z^2$: $\therefore x=(z^2-a^2)^{\frac{1}{2}}$, and $dx=\frac{zdz}{(z^2-a^2)^{\frac{1}{2}}}$:

$$\therefore \quad \int\frac{x^3dx}{(a^2+x^2)^{\frac{1}{2}}}=\int(z^2-a^2)dz=\frac{z^3}{3}-a^2z$$

$$=(a^2+x^2)^{\frac{1}{2}}\,\frac{x^2-2a^2}{3}.$$

Ex. 3. $\int\frac{x^2dx}{(a^2+x^2)^{\frac{5}{2}}}$. In this case, $m=2$, $n=2$, $\frac{p}{q}=-\frac{5}{2}$, and $\frac{m+1}{n}+\frac{p}{q}=-1$, a negative integer.

$$\int\frac{x^2dx}{(a^2+x^2)^{\frac{5}{2}}}=\int x^{-3}(1+a^2x^{-2})^{-\frac{5}{2}}dx.$$

Let $$1 + a^2x^{-2} = z^2: \therefore x = \frac{a}{(z^2-1)^{\frac{1}{2}}},$$

$$dx = -\frac{azdz}{(z^2-1)^{\frac{3}{2}}}:$$

$$\therefore \quad \int \frac{x^2dx}{(a^2+x^2)^{\frac{5}{2}}} = -\frac{1}{a^2}\int\frac{dz}{z^4} = \frac{1}{3a^2}\frac{1}{z^3}$$

$$= \frac{x^3}{3a^2(a^2+x^2)^{\frac{3}{2}}}.$$

Ex. 4. $\displaystyle\int \frac{dx}{x^2(1+x^2)^{\frac{1}{2}}}$. Here $m = -2$, $n = 2$, $\dfrac{p}{q} = -\dfrac{1}{2}$.

Put $1 + x^{-2} = z^2$: $\therefore x = \dfrac{1}{(z^2-1)^{\frac{1}{2}}}$, $dx = -\dfrac{zdz}{(z^2-1)^{\frac{3}{2}}}$:

$$\therefore \int \frac{dx}{x^2(1+x^2)^{\frac{1}{2}}} = \int \frac{dx}{x^3(1+x^{-2})^{\frac{1}{2}}} = -\int dz = -\frac{\sqrt{1+x^2}}{x}.$$

Functions in which the only irrational parts are monomials can always be rationalized and integrated. Thus, suppose it is required to find

$$\int \frac{\left(1 + x^{\frac{1}{2}} - x^{\frac{2}{3}}\right)dx}{1 + x^{\frac{1}{3}}}.$$

Put $x = t^6$; $\therefore dx = 6t^5dt$; and we have

$$\int \frac{\left(1 + x^{\frac{1}{2}} - x^{\frac{2}{3}}\right)dx}{1 + x^{\frac{1}{3}}} = \int \frac{(1+t^3-t^4)6t^5dt}{1+t^2}$$

$$= \int 6\left(-t^7 + t^6 + t^5 - t^4 + t^2 - 1 + \frac{1}{1+t^2}\right)dt$$

$$= -\frac{3}{4}t^8 + \frac{6}{7}t^7 + t^6 - \frac{6}{5}t^5 + 2t^3 - 6t + 6\tan.^{-1}t,$$

which becomes the integral in terms of x by replacing t by $x^{\frac{1}{6}}$.

The rule to be observed for rationalizing such expressions is to substitute for the quantity under the radical sign a new variable affected with the least common multiple of the indices of the radicals for its exponent.

Fractions in which the only radicals are the roots of the same binomial of the first degree may be reduced to the case just treated.

For example, required

$$\int \frac{x^2 + (ax+b)^{\frac{2}{3}} dx}{x + (ax+b)^{\frac{1}{2}}}.$$

Assume $\quad ax + b = t^6 : \therefore x = \dfrac{t^6 - b}{a}, \; dx = \dfrac{6t^5 dt}{a},$

$$(ax+b)^{\frac{2}{3}} = t^4, \; (ax+b)^{\frac{1}{2}} = t^3.$$

By these substitutions, the expression to be integrated becomes the rational fraction

$$\frac{6}{a^2} \frac{t^5\{(t^6 - b)^2 + a^2 t^4\} dt}{t^6 - b + at^3}.$$

The general method of integrating rational fractions will be investigated in the next section.

EXAMPLES.

1. $\displaystyle\int \frac{dx}{\sqrt{1 - 3x - x^2}} = \sin.^{-1} \frac{3 + 2x}{\sqrt{13}}.$

2. $\displaystyle\int x^n lx dx = \frac{x^{n+1}}{n+1}\left(lx - \frac{1}{n+1}\right).$

3. $\displaystyle\int \theta \sin.\theta d\theta = \sin.\theta - \theta \cos.\theta.$

4. $\displaystyle\int \frac{dx}{e^x + e^{-x}} = \tan.^{-1} e^x.$

5. $\displaystyle\int (1 - \cos.x)^2 dx = \frac{3}{2}x - 2\sin.x + \frac{\sin.2x}{4}.$

6. $$\int \frac{x^2 dx}{a^6 - x^6} = \frac{1}{6a^3}\, l\, \frac{a^3 + x^3}{a^3 - x^3}.$$

7. $$\int \frac{1 + \cos. x}{x + \sin. x} dx = l(x + \sin. x).$$

8. $$\int \frac{x + \sin. x}{1 + \cos. x} dx = x \tan. \frac{x}{2}.$$

9. $$\int \frac{l(lx)}{x} dx = lx\, l(lx) - lx.$$

10. $$\int e^{ax} \sin. mx \cos. nx dx$$

$$= \frac{e^{ax}}{2} \frac{a \sin.(m+n)x - (m+n)\cos.(m+n)x}{a^2 + (m+n)^2}$$

$$+ \frac{e^{ax}}{2} \frac{a \sin.(m-n)x - (m-n)\cos.(m-n)x}{a^2 + (m-n)^2}.$$

Having found the indefinite integral, the definite integral between assigned limits, except in special cases, can be at once determined.

11. $$\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{\pi a^2}{4};$$

for $$\int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin.^{-1} \frac{x}{a} = \psi(x),$$

and $$\psi(a) = \frac{\pi a^2}{4}, \quad \psi(0) = 0: \;\therefore\; \psi(a) - \psi(0) = \frac{\pi a^2}{4}.$$

12. $$\int_0^{2a} \text{ver.}^{-1} \frac{x}{a}\, dx = \pi a.$$

By making $x = a(1 - \cos. \theta)$, we find

$$\int \text{ver.}^{-1} \frac{x}{a} dx = \int a\theta \sin. \theta d\theta$$

$$= a \sin. \theta - a\theta \cos. \theta.$$

The limits π and 0 for the transformed integral correspond to the limits $2a$ and 0 for the given integral.

13. $$\int_0^{2a} x \operatorname{ver.}^{-1}\frac{x}{a}\,dx = \frac{5\pi a^2}{4}.$$

14. $$\int\frac{dx}{\sin. x+\cos. x}=\frac{1}{\sqrt{2}}\,l\tan.\left(\frac{x}{2}+\frac{\pi}{8}\right).$$

15. $$\int\frac{\sin.^2 x dx}{a+b\cos.^2 x}=\sqrt{\frac{a+b}{ab^2}}\tan.^{-1}\frac{\sqrt{a}\tan. x}{\sqrt{a+b}}-\frac{x}{b}.$$

16. $$\int x^3\sqrt{a+bx^2}\,dx=\left(\frac{a+bx^2}{5b^2}-\frac{a}{3b^2}\right)(a+bx^2)^{\frac{3}{2}}.$$

17. $$\int\frac{2a+x}{a+x}\sqrt{\frac{a-x}{a+x}}\,dx=\sqrt{a^2-x^2}-\frac{2a\sqrt{a-x}}{\sqrt{a+x}}.$$

18. $$\int\frac{(a+bx^n)^{\frac{3}{4}}dx}{x}=\frac{4}{3n}(a+bx^n)^{\frac{3}{4}}$$

$$+\frac{\sqrt[4]{a^3}}{n}\,l\,\frac{(a+bx^n)^{\frac{1}{4}}-a}{(a+bx^n)^{\frac{1}{4}}+a}+2\sqrt[4]{a^3}\tan.^{-1}\frac{(a+bx^n)^{\frac{1}{4}}}{\sqrt[4]{a}}.$$

In effecting this integration, transform by assuming

$$a+bx^n=z^4.$$

SECTION II.

INTEGRATION OF RATIONAL FRACTIONS BY DECOMPOSITION INTO PARTIAL FRACTIONS.

203. A RATIONAL fraction is of the form

$$\frac{A + Bx + Cx^2 + \cdots + Mx^m}{A' + B'x + C'x^2 + \cdots + N'x^n},$$

in which the numerator and denominator are entire and algebraic functions of x; the co-efficients A, $B\ldots$, A', $B'\ldots$, being constants.

Denote the numerator of such a fraction by $F(x)$, and the denominator by $f(x)$. If the degree, with respect to x, of $F(x)$, is not less than that of $f(x)$, we may divide $F(x)$ by $f(x)$ until we arrive at a remainder of a degree inferior to that of $f(x)$. Let $\varphi(x)$ be this remainder, and Q the quotient; then

$$\frac{F(x)}{f(x)} = Q + \frac{\varphi(x)}{f(x)},$$

$$\text{and } \int \frac{F(x)dx}{f(x)} = \int Qdx + \int \frac{\varphi(x)dx}{f(x)}.$$

As $\int Qdx$ can always be found, the integration of the original fraction is reduced to the integration of $\frac{\varphi(x)dx}{f(x)}$, in which the degree of $\varphi(x)$ is lower than that of $f(x)$. The integration of $\frac{\varphi(x)dx}{f(x)}$ is effected by resolving it into a series of more simple fractions, called *partial fractions;* and we will now demonstrate the possibility of such resolution in all cases in

which $f(x)$ can be separated into its factors of the first degree in respect to x.

204. Suppose the fraction $\dfrac{F(x)}{f(x)}$ to be in its lowest terms, and that the degree of $F(x)$ is inferior to that of $f(x)$. If the factor $x-a$ enters $f(x)$ p times, we shall have

$$f(x)=(x-a)^p\varphi(x),$$

$\varphi(x)$ denoting the product of the other factors of $f(x)$: whence

$$\frac{F(x)}{f(x)}=\frac{F(x)}{(x-a)^p\varphi(x)}=\frac{F(x)-\dfrac{F(a)}{\varphi(a)}\varphi(x)}{(x-a)^p\varphi(x)}+\frac{\dfrac{F(a)}{\varphi(a)}}{(x-a)^p}.$$

But $F(x)-\dfrac{F(a)}{\varphi(a)}\varphi(x)=0$ when $x=a$, and is therefore divisible by $x-a$. Let $\psi_1(x)$ be the quotient, then

$$\frac{F(x)}{f(x)}=\frac{\psi_1(x)}{(x-a)^{p-1}\varphi(x)}+\frac{F(a)}{\varphi(a)}\,\frac{1}{(x-a)^p}.$$

Denoting $\dfrac{F(a)}{\varphi(a)}$ by A_1, we have

$$\frac{F(x)}{f(x)}=\frac{\psi_1(x)}{(x-a)^{p-1}\varphi(x)}+\frac{A_1}{(x-a)^p};$$

that is, $\dfrac{F(x)}{f(x)}$ has been resolved into two parts, one of which is $\dfrac{A_1}{(x-a)^p}$. In like manner, $\dfrac{\psi_1(x)}{(x-a)^{p-1}\varphi(x)}$ may be reduced to

$$\frac{\psi_1(x)}{(x-a)^{p-1}\varphi(x)}=\frac{\psi_2(x)}{(x-a)^{p-2}\varphi(x)}+\frac{A_2}{(x-a)^{p-1}},$$

and so on, until at last we should have

$$\frac{\psi_{p-1}(x)}{(x-a)\varphi(x)}=\frac{\psi_p(x)}{\varphi(x)}+\frac{A_p}{x-a}.$$

By the successive substitution of these values of

$\frac{\psi_1(x)}{(x-a)^{p-1}\varphi(x)}, \frac{\psi_2(x)}{(x-a)^{p-2}\varphi(x)}\ldots$, in the order the inverse of that in which they were deduced, we find

$$\frac{F(x)}{f(x)} = \frac{\psi_p(x)}{\varphi(x)} + \frac{A_1}{(x-a)^p} + \frac{A_2}{(x-a)^{p-1}} + \cdots + \frac{A_p}{x-a}.$$

Proceeding in the same way with the fraction $\frac{\psi_p(x)}{\varphi(x)}$, the decomposition of $\frac{F(x)}{f(x)}$ into partial fractions will be at last completely effected.

205. If the root a is imaginary, and equal to $\alpha + \beta\sqrt{-1}$, then $a_1 = \alpha - \beta\sqrt{-1}$ is also a root of the equation $f(x) = 0$. Suppose that all the partial fractions corresponding to the real roots have been determined, and that there remains for resolution into such fractions the fraction $\frac{F_1(x)}{f_1(x)}$, in which $f_1(x) = 0$ gives rise to imaginary roots alone. Suppose, further, that the pair of roots, $x = \alpha + \beta\sqrt{-1}$, $x = \alpha - \beta\sqrt{-1}$, enters this equation q times. Denote the factor of $f_1(x)$ that gives the remaining imaginary roots by $\varphi_1(x)$; and, to abridge, make $a = \alpha + \beta\sqrt{-1}$, $a_1 = \alpha - \beta\sqrt{-1}$: then

$$\frac{F_1(x)}{f_1(x)} = \frac{F_1(x) - \frac{F_1(a)}{\varphi_1(a)}\varphi_1(x)}{(x-a)^q(x-a_1)^q\varphi_1(x)} + \frac{\frac{F_1(a)}{\varphi_1(a)}}{(x-a)^q(x-a_1)^q}$$

$$= \frac{\dfrac{F_1(x) - \frac{F_1(a)}{\varphi_1(a)}\varphi(x)}{x-a}}{(x-a)^{q-1}(x-a_1)^q} + \frac{\frac{F_1(a)}{\varphi_1(a)}}{(x-a)^q(x-a_1)^q} \quad (1).$$

In like manner,

$$\frac{\dfrac{F_1(x)-\dfrac{F_1(a)}{\varphi_1(a)}\varphi_1(x)}{x-a}}{(x-a)^{q-1}(x-a)^q\varphi_1(x)}$$

$$=\frac{\dfrac{F_1(x)-\dfrac{F_1(a)}{\varphi_1(a)}\varphi_1(x)}{x-a}-\dfrac{F_1(a_1)-\dfrac{F_1(a)}{\varphi_1(a)}\varphi_1(a_1)}{(a_1-a)\varphi_1(a_1)}\varphi_1(x)}{(x-a)^{q-1}(x-a_1)^q\varphi_1(x)}$$

$$+\frac{\dfrac{F_1(a_1)-\dfrac{F_1(a)}{\varphi_1(a)}\varphi_1(a_1)}{(a_1-a)\varphi_1(a_1)}}{(x-a)^{q-1}(x-a_1)^q} \quad (2).$$

The last term in the second member of Eq. 2 may be written

$$\frac{\dfrac{F_1(a_1)}{\varphi_1(a_1)}-\dfrac{F_1(a)}{\varphi_1(a)}}{(a_1-a)(x-a)^{q-1}(x-a)^q}.$$

But $(a_1-a)=-2\beta\sqrt{-1}$, and $\dfrac{F_1(a_1)}{\varphi_1(a_1)}$ is derived from $\dfrac{F_1(a)}{\varphi_1(a)}$ by changing the sign of $\sqrt{-1}$: hence, if $\dfrac{F_1(a)}{\varphi_1(a)}=A+B\sqrt{-1}$, then $\dfrac{F_1(a_1)}{\varphi_1(a_1)}=A-B\sqrt{-1}$, and

$$\frac{F_1(a_1)}{\varphi_1(a_1)}-\frac{F_1(a)}{\varphi_1(a)}=-2B\sqrt{-1}.$$

Therefore

$$\frac{\dfrac{F_1(a_1)}{\varphi_1(a_1)}-\dfrac{F_1(a)}{\varphi_1(a)}}{(a_1-a)(x-a)^{q-1}(x-a_1)^q}=\frac{\dfrac{B}{\beta}}{(x-a)^{q-1}(x-a_1)^q},$$

and the sum of the two partial fractions in the second members of Eqs. 1 and 2 will become

$$\frac{A+B\sqrt{-1}}{(x-a)^q(x-a_1)^q}+\frac{\frac{B}{\beta}}{(x-a)^{q-1}(x-a_1)^q}$$

$$=\frac{A+B\sqrt{-1}+\frac{B}{\beta}\left\{(x-\alpha)-\beta\sqrt{-1}\right\}}{(x-a)^q(x-a_1)^q}$$

$$=\frac{\frac{B}{\beta}(x-\alpha)+A}{(x^2-2\alpha x+\alpha^2+\beta^2)^q}.$$

which is rational. It is also seen, that the numerator and denominator of the first term in the second member of Eq. 2 is divisible by $x-a$. Dividing, and denoting the numerator of the result by $\chi_1(x)$, this term may be put under the form $\frac{\chi_1(x)}{(x^2-2\alpha x+\alpha^2+\beta^2)^{q-1}\varphi_1(x)}$: hence, by substitution in Eq. 1, we should have

$$\frac{F_1(x)}{f_1(x)}=\frac{\chi_1(x)}{(x^2-2\alpha x+\alpha^2+\beta^2)^{q-1}\varphi_1(x)}+\frac{\frac{B}{\beta}(x-\alpha)+A}{(x^2-2\alpha x+\alpha^2+\beta^2)^q}.$$

Now, $\chi_1(x)$ is a rational and entire function of x, and the fraction $\frac{\chi_1(x)}{(x^2-2\alpha x+\alpha^2+\beta^2)^{q-1}\varphi_1(x)}$ may be treated as was $\frac{F_1(x)}{f_1(x)}$, and so on; our result with respect to the assumed pair of imaginary roots being of the form

$$\frac{F_1(x)}{f_1(x)}=\frac{\chi_q(x)}{\varphi_1(x)}+\frac{M_1x+N_1}{(x^2-2\alpha x+\alpha^2+\beta^2)^q}+\cdots$$
$$+\frac{M_qx+N_q}{x^2-2\alpha x+\alpha^2+\beta^2}.$$

The possibility is thus demonstrated of resolving a fraction, the terms of which are rational functions of a single variable, into a series of rational partial fractions whenever the denominator of the given fraction can be separated into factors,

whether real or imaginary, of the first degree with respect to the variable. The investigation also shows the form of the partial fractions answering to the different kinds of factors of the denominator of the given fraction.

Thus if $f(x)$, the denominator of $\dfrac{F(x)}{f(x)}$, contains the factors

$$x-a,\ (x-b)^m,\ (x-c)^n,\ (x-\alpha-\beta\sqrt{-1})^p,$$

$$(x-\alpha+\beta\sqrt{-1})^p,\ x-\gamma-\delta\sqrt{-1},\ x-\gamma+\delta\sqrt{-1},$$

then

$$\frac{F(x)}{f(x)}=\frac{A}{x-a}+\frac{B_1}{(x-b)^m}+\frac{B_2}{(x-b)^{m-1}}+\cdots+\frac{B_m}{x-b}$$

$$+\frac{C_1}{(x-c)^n}+\frac{C_2}{(x-c)^{n-1}}+\cdots+\frac{C_n}{x-c}$$

$$+\frac{M_1x+N_1}{(x^2-2\alpha x+\alpha^2+\beta^2)^p}+\frac{M_2x+N_2}{(x^2-2\alpha x+\alpha^2+\beta^2)^{p-1}}$$

$$+\cdots+\frac{M_p+N_p}{x^2-2\alpha x+\alpha^2+\beta^2}+\frac{Px+Q}{x^2-2\gamma\delta+\gamma^2+\delta^2}.$$

The labor of determining the constants A_1, $A_2 \ldots B_1 \ldots M_1$, $N_1 \ldots$, for the partial fractions, which, by following the method above indicated, would be very great in many cases, may be diminished by expedients which we will now investigate. The most obvious of these is based on the consideration, that, when the partial fractions are reduced to a common denominator, the numerator of the result is identically equal to the numerator of the given fraction.

206. *To determine the partial fraction corresponding to the single real factor, $x-a$, of $f(x)$.*

Assume $$\frac{F(x)}{f(x)}=\frac{A}{x-a}+\frac{\psi(x)}{\varphi(x)} \quad (1),$$

in which A is a constant, and $\frac{\psi(x)}{\varphi(x)}$ is the sum of the partial fractions answering to the remaining factors of $f(x)$, and $f(x) = (x - a)\varphi(x)$.

From (1) we have

$$F(x) = A\varphi(x) + (x - a)\psi(x) \quad (2),$$

an identical equation. Make $x = a$, and then

$$F(a) = A\varphi(a): \therefore A = \frac{F(a)}{\varphi(a)}.$$

We also have the identical equation

$$f(x) = (x - a)\varphi(x);$$

whence, by differentiation,

$$f'(x) = \varphi(x) + (x - a)\varphi'(x),$$

an equation also identical: therefore, making $x = a$,

$$f'(a) = \varphi(a): \therefore A = \frac{F(a)}{f'(a)}.$$

207. *To determine the partial fractions corresponding to the real factor* $(x - a)$ *repeated n times.*

We now assume

$$\frac{F(x)}{f(x)} = \frac{A_1}{(x-a)^n} + \frac{A_2}{(x-a)^{n-1}} + \cdots \frac{A_n}{x-a} + \frac{\psi(x)}{\varphi(x)} \quad (1);$$

$\frac{\psi(x)}{\varphi(x)}$ denoting the sum of the fractions to which the other factors of $f(x)$ give rise.

Multiply both members of (1) by $(x - a)^n$, and we have the identical equation,

$$\frac{F(x)}{\varphi(x)} = A_1 + A_2(x-a) + \cdots + A_n(x-a)^{n-1} + \frac{\psi(x)}{\varphi(x)}(x-a)^n,$$

observing that $f(x) = \varphi(x)(x - a)^n$. Denoting the first mem-

ber of this equation by $\chi(x)$, and then, in it and its successive differential equations, making $x = a$, we have

$$\chi(a) = A_1,\ \chi'(a) = A_2,\ \chi''(a) = 1.2A_3, \ldots,$$
$$\chi^{(n-1)}(a) = 1.2\ldots(n-1)A_n,$$

and thus the numerators of the partial fractions are determined.

208. *To find the partial fraction corresponding to a single pair of imaginary factors.*

Suppose $x - \alpha - \beta\sqrt{-1}$, $x - \alpha + \beta\sqrt{-1}$, to be the imaginary factors of $f(x)$. We then put

$$\frac{F(x)}{f(x)} = \frac{Mx + N}{x^2 - 2\alpha x + \alpha^2 + \beta^2} + \frac{\psi(x)}{\varphi(x)}:$$

whence $F(x) = \varphi(x)(Mx + N) + \psi(x)(x^2 - 2\alpha x + \alpha^2 + \beta^2)$, an identical equation.

Make $x = \alpha + \beta\sqrt{-1}$; then

$$F(\alpha + \beta\sqrt{-1}) = \varphi(\alpha + \beta\sqrt{-1})\left\{M(\alpha + \beta\sqrt{-1}) + N\right\};$$

or, by making $x = \alpha - \beta\sqrt{-1}$,

$$F(\alpha - \beta\sqrt{-1}) = \varphi(\alpha - \beta\sqrt{-1})\left\{M(\alpha - \beta\sqrt{-1}) + N\right\}.$$

These last equations may be written

$$A + B\sqrt{-1} = (C + D\sqrt{-1})\left\{M(\alpha + \beta\sqrt{-1}) + N\right\},$$
$$A - B\sqrt{-1} = (C - D\sqrt{-1})\left\{M(\alpha - \beta\sqrt{-1}) + N\right\},$$

in which A, B, C, and D are known functions of α and β. From either of these equations, the values of M and N may be found by equating the real part of one member with the real part of the other, and the imaginary part of one member with the imaginary part of the other.

The values of M and N may also be found by the method of Art. 206. Thus, for brevity, denote the imaginary factors by $x - a$, $x - a_1$; then the partial fractions are

$$\frac{F(a)}{f'(a)}\,\frac{1}{x-a},\quad \frac{F(a_1)}{f'(a_1)}\,\frac{1}{x-a_1}.$$

If $\dfrac{F(a)}{f'(a)} = A + B\sqrt{-1}$, then $\dfrac{F(a_1)}{f'(a_1)} = A - B\sqrt{-1}$, since $\dfrac{F(a_1)}{f'(a_1)}$ is derived from $\dfrac{F(a)}{f'(a)}$ by changing the sign of $\sqrt{-1}$: hence, replacing a and a_1 by their values

$$\alpha + \beta\sqrt{-1},\ \alpha - \beta\sqrt{-1},$$

the fractions become

$$\frac{A + B\sqrt{-1}}{x - \alpha - \beta\sqrt{-1}},\ \frac{A - B\sqrt{-1}}{x - \alpha + \beta\sqrt{-1}},$$

the sum of which is

$$\frac{2A(x - \alpha) + 2B\beta}{x^2 - 2\alpha x + \alpha^2 + \beta^2}.$$

209. *To find the partial fractions corresponding to a pair of imaginary factors which enters the denominator of the given fraction several times.*

Let $x - \alpha - \beta\sqrt{-1}$, $x - \alpha + \beta\sqrt{-1}$, be the imaginary factors, and, to abridge, put $a = \alpha + \beta\sqrt{-1}$, $a_1 = \alpha - \beta\sqrt{-1}$; then, putting $f(x) = \{(x - a)(x - a_1)\}^q \varphi(x)$,

$$\frac{F(x)}{f(x)} = \frac{M_1 x + N_1}{\{(x - a)(x - a_1)\}^q} + \frac{M_2 x + N_2}{\{(x - a)(x - a_1)\}^{q-1}} + \cdots$$
$$+ \frac{M_q x + N_q x}{\{(x - a)(x - a_1)\}} + \frac{\psi(x)}{\varphi(x)},$$

representing by $\dfrac{\psi(x)}{\varphi(x)}$ the sum of the partial fractions to which the remaining factors of $f(x)$ give rise. Multiply the first member of this equation by $f(x)$, and the second member by its equal $\{(x - a)(x - a_1)\}^q \varphi(x)$, and we have

$$F(x) = (M_1 x + N_1)\varphi(x) + (M_2 x + N_2)(x - a)(x - a_1)\varphi(x)$$
$$+ (M_3 x + N_3)\{(x - a)(x - a_1)\}^2 \varphi(x) + \cdots$$
$$+ \{(x - a)(x - a_1)\}^q \varphi(x) \quad (1).$$

Now, whether we make $x = a$, or $x = a_1$, all the terms in the second member of this equation, except the first term, vanish. Suppose $x = a$, then

$$F(a) = (M_1 a + N_1)\varphi(a);$$

and if the real parts in the two members of this last be equated, and also the imaginary parts, we shall have two equations from which to find the values of M_1 and N_1. Substitute these values in (1), transpose $(M_1 x + N_1)\varphi(x)$, and divide through by $(x-a)(x-a_1)$, denoting the first member of the resulting equation by $F_1(x)$; then

$$F_1(x) = (M_2 x + N_2)\varphi(x) + (M_3 x + N_3)(x-a)(x-a_1)\varphi(x) + \cdots$$
$$+ \{(x-a)(x-a_1)\}^{q-1}\varphi(x) \quad (2).$$

Proceeding with (2) as we did with (1), the values of M_2 and N_2 may be found; and, by repeating these operations, all of the constants, M_1, N_1, M_2, $N_2 \ldots$, will finally be determined.

210. The rational fraction, which may be decomposed into partial fractions by the foregoing methods, being a differential co-efficient, the resulting fractions are also differential co-efficients; and the sum of their integrals will be the integral of the given fraction.

The differentials corresponding to these partial fractions are of the form

$$\frac{Adx}{(x-a)^m}, \quad \frac{(Mx+N)dx}{(x^2+px+q)^n}.$$

The integral of the former is $-\dfrac{1}{m-1}\,\dfrac{A}{(x-a)^{m-1}}$, which becomes $Al(x-a)$ when $m = 1$; and that of the latter, when $n = 1$, has been explained in Ex. 20, Art. 201. The integration of the second form, if n be greater than unity, is reserved for the next section.

EXAMPLES.

1. $\frac{(3-2x)dx}{x^2-x-2}=\frac{F(x)}{f(x)}$. The factors of the denominator are $x+1$, $x-2$. We therefore put

$$\frac{3-2x}{x^2-x-2}=\frac{A}{x+1}+\frac{A_1}{x-2}.$$

Substituting -1 and $+2$ successively for x in

$$\frac{F(x)}{f'(x)}=\frac{3-2x}{2x-1},$$

we get $$A=-\frac{5}{3},\ A_1=-\frac{1}{3}:$$

$$\therefore \qquad \frac{(3-2x)dx}{x^2-x-2}=-\frac{5}{3}\frac{dx}{x+1}-\frac{1}{3}\frac{dx}{x-2},$$

$$\int\frac{(3-2x)dx}{x^2-x-2}=-\frac{5}{3}l(x+1)-\frac{1}{3}l(x-2).$$

2. $\int\frac{(x^2-3x-2)dx}{(x^2+x+1)^2(x+1)}$.

In the denominator of this, the pair of imaginary factors, $x+\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, $x+\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, enters twice; and the real factor $x+1$ once. We put

$$\frac{x^2-3x-2}{(x^2+x+1)^2(x+1)}=\frac{M_1x+N_1}{(x^2+x+1)^2}+\frac{M_2x+N_2}{x^2+x+1}+\frac{\psi(x)}{x+1}:$$

$$\therefore \qquad x^2-3x-2=(M_1x+N_1)(x+1)$$
$$+(M_2x+N_2)(x^2+x+1)(x+1)+(x^2+x+1)^2\psi(x) \quad (1).$$

Give to x one of the values which reduce x^2+x+1 to zero; then, for this value, (1) becomes

$$x^2-3x-2=(M_1x+N_1)(x+1) \quad (2).$$

From $x^2+x+1=0$, we have $x^2=-x-1$. Substituting in (2),

$$\begin{aligned}-4x-3 &= M_1x^2+M_1x+N_1x+N_1\\ &= M_1(-x-1)+M_1x+N_1x+N_1\\ &= -M_1+N_1x+N_1:\end{aligned}$$

whence $\qquad M_1-N_1=3,\ N_1=-4,\ M_1=-1.$

In (1), replacing M_1 and N_1 by the values thus found, and transposing, we have

$$x^2-3x-2+(x+4)(x+1)=2(x^2+x+1)$$
$$=(M_2x+N_2)(x^2+x+1)(x+1)+(x^2+x+1)^2\psi(x) \quad (3).$$

Dividing through by x^2+x+1, and in the result making $x^2+x+1=0$, we get

$$2=(M_2x+N_2)(x+1):$$

whence $\qquad M_2=-2,\ N_2=0.$

The partial fraction corresponding to $\dfrac{\psi(x)}{x+1}$ may be found by the method explained in Art. 206; or thus: After dividing (3) by x^2+x+1, replace M_2 and N_2 by their values, transpose, and again divide by x^2+x+1. We find $\psi(x)=2$:

therefore $$\int\frac{(x^2-3x-2)dx}{(x^2+x+1)^2(x+1)}=-\int\frac{(x+4)dx}{(x^2+x+1)^2}-\int\frac{2xdx}{x^2+x+1}+\int\frac{2dx}{x+1}.$$

3. $\displaystyle\int\frac{(9x^2+9x-128)dx}{x^3-5x^2+3x+9}.$

By the method of equal roots, we readily discover that the denominator may be resolved into the factors $(x-3)^2$, $x+1$: hence we put

$$\frac{9x^2+9x-128}{x^3-5x^2+3x+9}=\frac{A_1}{(x-3)^2}+\frac{A_2}{x-3}+\frac{B_1}{x+1}:$$

whence

$$9x^2+9x-128=A_1(x+1)+A_2(x-3)(x+1)+B_1(x-3)^2;$$

from which, by making $x=3$ and $x=-1$ successively, we get $A_1=-5$, $B_1=-8$. If the second member were developed, the co-efficient of x^2 would be A_2+B_1; equating this with the co-efficient of x^2 in the first member, we have $A_2+B_1=9$: $\therefore A_2=17$; and therefore

$$\int\frac{(9x^2+9x-128)dx}{x^3-5x^2+3x+9}=-\int\frac{5dx}{(x-3)^2}+\int\frac{17dx}{x-3}-\int\frac{8dx}{x+1}$$

$$=\frac{5}{x-3}+17l(x-3)-8l(x+1).$$

211. Integration of $\dfrac{x^{m-1}dx}{x^n-1}$ when m and n are positive integers.

If n be an even number, the real roots of $x^n-1=0$ are $+1$ and -1; and the imaginary roots (Art. 77) are given by the expression $\cos.\dfrac{2k\pi}{n}\pm\sqrt{-1}\sin.\dfrac{2k\pi}{n}$, by giving to k in succession the values $1, 2, 3\ldots, \dfrac{n}{2}-1$.

We will denote the arc $\dfrac{\pi}{n}$ by θ, $\dfrac{F(x)}{f(x)}$ being the fraction to to be resolved into partial fractions. It has been shown (Art. 206), that, if a be a root of $f(x)=0$, the corresponding partial fraction is $\dfrac{F(a)}{f'(a)}\dfrac{1}{x-a}$: hence, for the fraction $\dfrac{x^{m-1}}{x^n-1}$, the partial fraction for the root $+1$ of the equation $x^n-1=0$ is $\dfrac{a^{m-1}}{na^{n-1}}=\dfrac{a^m}{na^n}=\dfrac{1}{n(x-1)}$; and, for the root -1, the partial fraction is $\dfrac{(-1)^m}{n(x+1)}$. The pair of imaginary roots,

$$\cos. 2k\theta\pm\sqrt{-1}, \sin. 2k\theta,$$

give the partial fractions

$$\frac{(\cos. 2k\theta + \sqrt{-1}\sin. 2k\theta)^m}{n(x - \cos. 2k\theta - \sqrt{-1}\sin. 2k\theta)} + \frac{(\cos. 2k\theta - \sqrt{-1}\sin. 2k\theta)^m}{n(x - \cos. 2k\theta + \sqrt{-1}\sin. 2k\theta)};$$

that is, (Art. 73),

$$\frac{\cos. 2mk\theta + \sqrt{-1}\sin. 2mk\theta}{n(x - \cos. 2k\theta - \sqrt{-1}\sin. 2k\theta)} + \frac{\cos. 2mk\theta - \sqrt{-1}\sin. 2mk\theta}{n(x - \cos. 2k\theta + \sqrt{-1}\sin. 2k\theta)}$$

$$= \frac{2\cos. 2mk\theta(x - \cos. 2k\theta) - 2\sin. 2mk\theta \sin. 2k\theta}{n(x^2 - 2x\cos. 2k\theta + 1)};$$

and for each pair of imaginary roots, that is, for each of the values 1, 2, 3..., $\frac{n}{2} - 1$ of k, there will be a partial fraction of the form of this. Let the symbol Σ denote the sum of these; then

$$\int\frac{x^{m-1}dx}{x^n - 1} = \int\frac{dx}{n(x-1)} + \int\frac{(-1)^m dx}{n(x+1)}$$
$$+ \frac{2}{n}\int\Sigma\frac{\cos. 2mk\theta(x - \cos. 2k\theta) - \sin. 2mk\theta \sin. 2k\theta}{(x - \cos. 2k\theta)^2 + \sin.^2 2k\theta}$$
$$= \frac{1}{n}l(x-1) + \frac{(-1)^m}{n}l(x+1)$$
$$+ \frac{1}{n}\Sigma\cos. 2mk\theta l(x^2 - 2x\cos. 2k\theta + 1)$$
$$- \frac{2}{n}\Sigma\sin. 2mk\theta \tan.^{-1}\frac{x - \cos. 2k\theta}{\sin. 2k\theta},$$

by observing that the last term under the sign of integration can be separated into the two fractions

$$\frac{2}{n}\Sigma\frac{\cos. 2mk\theta(x - \cos. 2k\theta)}{x^2 - 2x\cos. 2k\theta + 1} \text{ and } -\frac{2}{n}\frac{\sin. 2mk\theta \sin. 2k\theta}{(x - \cos. 2k\theta)^2 + \sin.^2 2k\theta}.$$

212. Integration of $\frac{x^{m-1}dx}{x^n - 1}$, m and n being positive integers, and n an odd number.

In this case, $x^n - 1 = 0$ has but one real root, $+1$; and the imaginary roots are the values assumed by the expression $\cos.\frac{2k\pi}{n} \pm \sqrt{-1}\sin.\frac{2k\pi}{n}$, by giving to k in succession the values $1, 2, 3\ldots, \frac{n-1}{2}$ (Art. 77). Hence, by operating as in the preceding article, we find

$$\int\frac{x^{m-1}dx}{x^n-1} = \frac{1}{n}l(x-1) + \frac{1}{n}\Sigma\cos.2mk\theta l(x^2 - 2x\cos.2k\theta + 1)$$
$$-\frac{2}{n}\Sigma\sin.2mk\theta\tan.^{-1}\frac{x-\cos.2k\theta}{\sin.2k\theta}.$$

213. Integration of $\frac{x^{m-1}dx}{x^n+1}$, m and n being entire and positive, and n even.

Under the supposition, none of the roots of $x^n + 1 = 0$ are real; and the imaginary roots are found by giving to k, in the expression $\cos.\frac{2k+1}{n}\pi \pm \sqrt{-1}\sin.\frac{2k+1}{n}\pi$, the values $0, 1, 2\ldots, \frac{n}{2} - 1$ in succession. Put θ for $\frac{\pi}{n}$, then the partial fractions corresponding to a pair of these roots will be

$$\frac{\cos.(2k+1)\theta + \sqrt{-1}\sin.(2k+1)\theta}{x - \cos.(2k+1)\theta - \sqrt{-1}\sin.(2k+1)\theta}$$
$$+\frac{\cos.(2k+1)\theta - \sqrt{-1}\sin.(2k+1)\theta}{x - \cos.(2k+1)\theta + \sqrt{-1}\sin.(2k+1)\theta},$$

the sum of which is

$$-\frac{2}{n}\frac{\cos.m(2k+1)\theta\{x - \cos.(2k+1)\theta\} - \sin.m(2k+1)\theta\sin.(2k+1)\theta}{\{x - \cos.(2k+1)\theta\}^2 + \sin.^2(2k+1)\theta}.$$

Hence

$$\int\frac{x^{m-1}dx}{x^n+1} = -\frac{1}{n}\Sigma\cos.m(2k+1)\theta l\{x^2 - 2x\cos.(2k+1)\theta + 1\}$$
$$+\frac{2}{n}\Sigma\sin.m(2k+1)\theta\tan.^{-1}\frac{x - \cos.(2k+1)\theta}{\sin.(2k+1)\theta}.$$

In like manner, we integrate $\frac{x^{m-1}dx}{x^n+1}$, when n is odd, by finding the partial fractions corresponding to the roots of $x^n+1=0$. In this case, there is one real root, -1; and the other roots, which are imaginary, are the values assumed by the expression

$$\cos.(2k+1)\frac{\pi}{n} \pm \sqrt{-1}\sin.(2k+1)\frac{\pi}{n},$$

by giving to k the values $1.2.3\ldots\frac{n-1}{2}$ successively.

We should find

$$\int\frac{x^{m-1}dx}{x^n+1} = \frac{(-1)^m}{n}l(x+1)$$

$$-\frac{1}{n}\Sigma\cos.m(2k+1)\theta\left\{x^2-2x\cos.(2k+1)\theta+1\right\}$$

$$+\frac{2}{n}\Sigma\sin.m(2k+1)\theta\tan.^{-1}\frac{x-\cos.(2k+1)\theta}{\sin.(2k+1)\theta}.$$

EXAMPLES.

1. $$\int\frac{dx}{x^3-1} = \frac{1}{6}\,l\,\frac{(x-1)^2}{x^2+x+1} - \frac{1}{\sqrt{3}}\tan.^{-1}\frac{2x+1}{\sqrt{3}}.$$

2. $$\int\frac{dx}{a^4-x^4} = \frac{1}{2a^3}\tan.^{-1}\frac{x}{a} + \frac{1}{4a^3}\,l\,\frac{a+x}{a-x}.$$

3. $$\int\frac{x^2dx}{x^4+x^2-2} = \frac{1}{6}\,l\,\frac{x-1}{x+1} + \frac{1}{3}\sqrt{2}\tan.^{-1}\frac{x}{\sqrt{2}}.$$

4. $$\int\frac{dx}{(x^2+a^2)(x+b)} = \frac{1}{b^2+a^2}\left(l\,\frac{x+b}{\sqrt{x^2+a^2}} + \frac{b}{a}\tan.^{-1}\frac{x}{a}\right).$$

5. $$\int\frac{x^2dx}{x^4+1} = \frac{1}{4}\sqrt{2}\,l\,\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}.$$

6. $$\int\frac{x^3dx}{x^6+1} = \frac{1}{12}l(x^4-x^2+1) - \frac{1}{6}l(x^2+1)$$

$$+\frac{1}{2}\sqrt{3}\left\{\tan.^{-1}(2x-\sqrt{3}) - \tan.^{-1}(2x+\sqrt{3})\right\}.$$

7. $\int \frac{dx}{(1-x^3)^{\frac{1}{3}}}$. This may be rationalized by putting

$$1 - x^3 = x^3 z^3;$$

whence $dx = -\frac{z^2 dz}{(z^3+1)^{\frac{4}{3}}}$, $(1-x^3)^{\frac{1}{3}} = \frac{z}{(1+z^3)^{\frac{1}{3}}}$:

$$\therefore \int \frac{dx}{(1-x^3)^{\frac{1}{3}}} = -\int \frac{z dz}{z^3+1}$$

$$= \frac{1}{3} l(z+1) - \frac{1}{6} l(z^2 - z + 1) - \frac{1}{3\sqrt{3}} \tan.^{-1} \frac{2z-1}{\sqrt{3}};$$

in which, if we replace z by its value $\frac{\sqrt[3]{1-x^3}}{x}$, we have the required integral in terms of x.

SECTION III.

FORMULÆ FOR THE INTEGRATION OF BINOMIAL DIFFERENTIALS BY SUCCESSIVE REDUCTION.

214. The integration of differentials of the form

$$x^m(a+bx^n)^p dx$$

may be made to depend on that of other expressions of the same form, in which the exponent of the variable without the parenthesis, or the exponent of the parenthesis itself, is less than in the original expression. This is accomplished by the method of integration by parts. We have

$$\int x^m(a+bx^n)^p dx = \int x^{m-n+1}(a+bx^n)^p x^{n-1}dx = \int u dv$$

by making $u = x^{m-n+1}$, $v = \dfrac{(a+bx^n)^{p+1}}{nb(p+1)}$;

and therefore

$$\int x^m(a+bx^n)^p dx = x^{m-n+1}\frac{(a+bx^n)^{p+1}}{nb(p+1)}$$

$$-\frac{m-n+1}{nb(p+1)}\int x^{m-n}(a+bx^n)^{p+1}dx \quad (1).$$

The integration of $x^m(a+bx^n)^p$ is thus brought to that of $x^{m-n}(a+bx^n)^{p+1}$, which last is more simple than the first when m is positive, and greater than n, and when p is negative; for then the numerical value of $p+1$ is less than p.

But we may find a formula in which the exponent of the variable without the parenthesis is diminished, while that of the parenthesis itself is unchanged.

Thus we have the identical equation

$$x^{m-n}(a+bx^n)^{p+1} = x^{m-n}(a+bx^n)^p(a+bx^n)$$

$$= ax^{m-n}(a+bx^n)^p + bx^m(a+bx^n)^p;$$

therefore

$$\int x^{m-n}(a+bx^n)^{p+1}dx = a\int x^{m-n}(a+bx^n)^p dx$$
$$+ b\int x^m(a+bx^n)^p dx.$$

Substituting this value in Eq. 1, we have

$$\int x^m(a+bx^n)^p dx = x^{m-n+1}\frac{(a+bx^n)^{p+1}}{nb(p+1)}$$
$$-\frac{m-n+1}{nb(p+1)}a\int x^{m-n}(a+bx^n)^p dx$$
$$-\frac{m-n+1}{nb(p+1)}b\int x^m(a+bx^n)^p dx;$$

whence, by transposition and reduction,

$$\int x^m(a+bx^n)^p dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(m+np+1)}$$
$$-\frac{a(m-n+1)}{b(m+np+1)}\int x^{m-n}(a+bx^n)^p dx \quad \text{(A).}$$

The integration of $x^m(a+bx^n)^p dx$ is then made to depend on that of $x^{m-n}(a+bx^n)^p dx$; and, by another application of the formula, the integration of this last reduces to that of $x^{m-2n}(a+bx^n)^p dx$, and so on: hence, if m is positive, and greater than n, and i denote the entire part of the quotient $\frac{m}{n}$, the integral to be determined after a number, i, of reductions, would be

$$\int x^{m-in}(a+bx^n)^p dx.$$

If $m-in=n-1$, this expression is immediately integrable; for

$$\int x^{n-1}(a+bx^n)^p dx = \frac{(a+bx^n)^{p+1}}{nb(p+1)};$$

but $m-in=n-1$ leads to $\frac{m+1}{n}=i+1$, and the condition of integrability (Art. 202) is then satisfied.

Formula A cannot be applied when $m + np + 1 = 0$; for then its second member takes the form $\infty - \infty$: but in this case $\frac{m+1}{n} + p$ is equal to zero, that is, an entire number; and the original expression is therefore immediately integrable.

215. Formula for the reduction of the exponent of the parenthesis.

Assume

$$x^m(a + bx^n)^p dx = (a + bx^n)^p d\frac{x^{m+1}}{m+1} = udv;$$

then, integrating by parts,

$$\int x^m(a + bx^n)^p dx = \frac{x^{m+1}(a + bx^n)^p}{m+1}$$
$$- \frac{pnb}{m+1}\int x^{m+n}(a + bx^n)^{p-1}dx \quad (1).$$

In this formula, the exponent of the binomial has been diminished by 1, while that of x without the parenthesis has been increased n units. We may, however, diminish the former without increasing the latter exponent. In formula A, last article, change m into $m + n$, and p into $p - 1$: we thus have

$$\int x^{m+n}(a + bx^n)^{p-1}dx = \frac{x^{m+1}(a + bx^n)^p}{b(np + m + 1)}$$
$$- \frac{(m+1)a}{b(np+m+1)}\int x^m(a+bx^n)^{p-1}dx;$$

and this value of $\int x^{m+n}(a + bx^n)^{p-1}dx$, substituted in Eq. 1, gives, after reduction,

$$\int x^m(a + bx^n)^p dx = \frac{x^{m+1}(a + bx^n)^p}{np + m + 1}$$
$$+ \frac{anp}{np+m+1}\int x^m(a + bx^n)^{p-1}dx \quad (B).$$

By the repeated application of this formula, the exponent p will be diminished by all the units it contains. This formula will not admit of application when $np+m+1=0$; but then the integral $\int x^m(a+bx^n)^p dx$ can be found at once (Art. 202). By means of formulæ A and B, the integral $\int x^m(a+bx^n)^p dx$, when m and p are positive, may be made to depend on the more simple integral $\int x^{m-in}(a+bx^n)^{p-q}dx$; in being the greatest multiple of n less than m, and q the entire part of p.

216. Formula for the reduction of the exponent m when m is negative.

From Eq. A, Art. 214, by transposition and division, we find

$$\int x^{m-n}(a+bx^n)^p dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{a(m-n+1)} - \frac{b(m+np+1)}{a(m-n+1)}\int x^m(a+bx^n)^p dx.$$

Changing $m-n$ into $-m$, this becomes

$$\int x^{-m}(a+bx^n)^p dx = -\frac{x^{-m+1}(a+bx^n)^{p+1}}{a(m-1)} + \frac{b(np+n-m+1)}{a(m-1)}\int x^{-m+n}(a+bx^n)^p dx \quad \text{(C)}.$$

If in denote the greatest multiple of n contained in m, then, by $i+1$ applications of this formula, the integration of

$$x^{-m}(a+bx^n)^p dx$$

will depend on that of $x^{-m+(i+1)n}(a+bx^n)^p dx$; and, if we have $-m+(i+1)n=n-1$,

we have $$\int x^{n-1}(a+bx^n)^p dx = \frac{(a+bx^n)^{p+1}}{nb(p+1)}.$$

But under this supposition, since $\frac{-m+1}{n}=-i$, an entire number, the original expression is immediately integrable (Art. 202).

217. Formula for the reduction of the exponent p when p is negative.

From Formula B (Art. 215), we find

$$\int x^m(a+bx^n)^{p-1}dx = -\frac{x^{m+1}(a+bx^n)^p}{anp}$$
$$+\frac{np+m+1}{anp}\int x^m(a+bx^n)^p dx;$$

and, if in this we replace $p-1$ by $-p$, it becomes

$$\int x^m(a+bx^n)^{-p}dx = \frac{x^{m+1}(a+bx^n)^{-p+1}}{an(p-1)}$$
$$-\frac{m+n+1-pn}{an(p-1)}\int x^m(a+bx^n)^{-p+1}dx \quad \text{(D)}.$$

By the continued application of this formula, the exponent of the binomial will finally be reduced to a positive proper fraction. When $p=1$, it cannot be applied; but then the integration of the given expression may be brought to that of a rational fraction.

218. The preceding formulæ facilitate the integration of binomial differentials; but it is to be observed that the examples to which they are applicable belong to cases of integrability before established (Art. 202), and the results may therefore be obtained independently.

By the application of Formula A, we have

1. $$\int\frac{x^m dx}{\sqrt{1-x^2}} = -\frac{x^{m-1}\sqrt{1-x^2}}{m}+\frac{m-1}{m}\int\frac{x^{m-2}dx}{\sqrt{1-x^2}};$$

and, by making $m = 1.3.5\ldots$ successively, this gives

$$\int\frac{xdx}{\sqrt{1-x^2}} = -\sqrt{1-x^2},$$

$$\int\frac{x^3 dx}{\sqrt{1-x^2}} = -\frac{x^2}{3}\sqrt{1-x^2}+\frac{2}{3}\int\frac{xdx}{\sqrt{1-x^2}},$$

$$\int \frac{x^5 dx}{\sqrt{1-x^2}} = -\frac{x^4}{5}\sqrt{1-x^2} + \frac{4}{5}\int \frac{x^3 dx}{\sqrt{1-x^2}}:$$

. . . .

whence

$$\int \frac{x dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2},$$

$$\int \frac{x^3 dx}{\sqrt{1-x^2}} = -\left(\frac{x^2}{3} + \frac{2}{1.3}\right)\sqrt{1-x^2},$$

$$\int \frac{x^5 dx}{\sqrt{1-x^2}} = -\left(\frac{x^4}{5} + \frac{4x^2}{3.5} + \frac{2.4}{1.3.5}\right)\sqrt{1-x^2}.$$

. . . .

When m is an odd number, we have the general formula

$$\int \frac{x^m dx}{\sqrt{1-x^2}}$$

$$= \left\{\frac{x^{m-1}}{m} + \frac{(m-1)x^{m-3}}{(m-2)m} + \cdots + \frac{2.4\ldots(m-1)}{1.3\ldots m}\right\}\sqrt{1-x^2};$$

and, when m is an even number,

$$\int \frac{x^m dx}{\sqrt{1-x^2}}$$

$$= \left\{\frac{x^{m-1}}{m} + \frac{(m-1)x^{m-3}}{(m-2)m} + \cdots + \frac{1.3.5\ldots(m-1)}{2.4.6\ldots n}x\right\}\sqrt{1-x^2}$$
$$+ \frac{1.3.5\ldots(m-1)}{2.4.6\ldots m}\sin.^{-1}x.$$

2. $$\int \frac{dx}{x^m(a^2+x^2)^{\frac{1}{2}}} = \int x^{-m}(a^2+x^2)^{-\frac{1}{2}}dx.$$

Comparing this with Formula C, and making

$$b = 1,\ n = 2,\ p = -\frac{1}{2},$$

we find

$$\int \frac{dx}{x^m(a^2+x^2)^{\frac{1}{2}}} = -\frac{(a^2+x^2)^{\frac{1}{2}}}{(m-1)a^2 x^{m-1}}$$
$$-\frac{m-2}{(m-1)a^2}\int \frac{dx}{x^{m-2}(a^2+x^2)^{\frac{1}{2}}}.$$

Without referring to the formula, this expression may be found as follows: —

$$\int \frac{dx}{x^m(a^2+x^2)^{\frac{1}{2}}} = \int \frac{d(a^2+x^2)^{\frac{1}{2}}}{dx} \cdot \frac{1}{x^{m+1}} dx$$

$$= \frac{(a^2+x^2)^{\frac{1}{2}}}{x^{m+1}} + (m+1)\int \frac{a^2+x^2}{x^{m+2}(a^2+x^2)^{\frac{1}{2}}} dx;$$

whence, by transposition and reduction,

$$(m+1)a^2\int \frac{dx}{x^{m+2}(a^2+x^2)^{\frac{1}{2}}} = \frac{(a^2+x^2)^{\frac{1}{2}}}{x^{m+1}} - m\int \frac{dx}{x^m(a^2+x^2)^{\frac{1}{2}}}.$$

$$\therefore \quad \int \frac{dx}{x^m(a^2+x^2)^{\frac{1}{2}}} = -\frac{(a^2+x^2)^{\frac{1}{2}}}{(m-1)a^2x^{m-1}}$$

$$-\frac{m-2}{(m-1)a^2}\int \frac{dx}{x^{m-2}(a^2+x^2)^{\frac{1}{2}}}.$$

219. By means of Formula D, the expression $\frac{(Mx+N)dx}{(x^2+px+q)^n}$, which occurs in Art. 210, may be integrated by successive reduction.

Let $\alpha+\beta\sqrt{-1}$, $\alpha-\beta\sqrt{-1}$, be the roots of the equation $x^2+px+q=0$; then

$$\frac{(Mx+N)dx}{(x^2+px+q)^n} = \frac{(Mx+N)dx}{\{(x-\alpha)^2+\beta^2\}^n}$$

$$= \frac{M(x-\alpha)dx}{\{(x-\alpha)^2+\beta^2\}^n} + (M\alpha+N)\frac{dx}{\{(x-\alpha)^2+\beta^2\}^n}.$$

Putting $x-\alpha=z$, we find

$$\int \frac{M(x-\alpha)dx}{\{(x-\alpha)^2+\beta^2\}^n} = \int Mz(z^2+\beta^2)^{-n}dz$$

$$= -\frac{M}{2(n-1)}\,\frac{1}{(z^2+\beta^2)^{n-1}}$$

$$= -\frac{M}{2(n-1)}\,\frac{1}{\{(x-\alpha)^2+\beta^2\}^{n-1}}.$$

Making $x - \alpha = y$, and $M\alpha + N = M'$, we have

$$\int \frac{(M\alpha + N)\,dx}{\{(x-\alpha)^2 + \beta^2\}^n} = \int \frac{M'\,dy}{(y^2+\beta^2)^n} = \int M'(y^2+\beta^2)^{-n}dy;$$

and therefore, by combining these results, we have

$$\int \frac{(Mx+N)\,dx}{\{(x-\alpha)^2+\beta^2\}^n} = -\frac{M}{2(n-1)}\,\frac{1}{\{(x-\alpha)^2+\beta^2\}^{n-1}} + \int M'(y^2+\beta^2)^{-n}dy.$$

Formula D may now be applied to the term under the sign $\int$ in the second member of this last equation; and, by repeated applications, the exponent $-n$ will be reduced to -1, when the integral will be completely determined.

220. Reduction formulæ may also be constructed to facilitate the integration of trigonometrical functions. Let the integral of $\sin.^p x \cos.^q x dx$ be required.

Make $\sin. x = z$; then

$$\cos. x = (1-z^2)^{\frac{1}{2}},\quad dx = (1-z^2)^{-\frac{1}{2}}dz,$$

and
$$\sin.^p x \cos.^q x dx = z^p(1-z^2)^{\frac{q-1}{2}}dz.$$

Now, if q be an odd number, whether positive or negative, we may always effect the integration of $z^p(1-z^2)^{\frac{q-1}{2}}dz$, whatever may be the value of p. In like manner, by making $\cos. x = z$, we see that the integration can be effected when p is an odd number, whether positive or negative, whatever be the value of q.

In any case in which $p > 0$, and $\frac{q-1}{2} > 0$, by applying Formulæ A and B, we get

$$\int z^p(1-z^2)^{\frac{q-1}{2}}dz = -\frac{z^{p-1}(1-z^2)^{\frac{q+1}{2}}}{p+q} + \frac{p-1}{p+q}\int z^{p-2}(1-z^2)^{\frac{q-1}{2}}dz \qquad (A'),$$

$$\int z^p(1-z^2)^{\frac{q-1}{2}}dz = \frac{z^{p+1}(1-z^2)^{\frac{q-1}{2}}}{p+q} + \frac{q-1}{p+q}\int z^p(1-z^2)^{\frac{q-3}{2}}dz \qquad (B').$$

If $p < 0$, Formula C gives

$$\int z^{-p}(1-z^2)^{\frac{q-1}{2}}dz = -\frac{z^{-p+1}(1-z^2)^{\frac{q+1}{2}}}{p-1} + \frac{p-q-2}{p-1}\int z^{-p+2}(1-z^2)^{\frac{q-1}{2}}dz \qquad (C');$$

and when $\frac{q-1}{2} < 0$, by Formula D, we have

$$\int z^p(1-z^2)^{-\frac{q-1}{2}}dz = \frac{z^{p+1}(1-z^2)^{-\frac{q-3}{2}}}{q-3} - \frac{p-q+4}{q-3}\int z^p(1-z^2)^{-\frac{q-3}{2}}dz \qquad (D').$$

By the aid of the foregoing formulæ, we are enabled to make the integration of $\sin.^p x \cos.^q x dx$ depend on that of expressions in which the exponents p and q are numerically less than in the original function. From (A′) we have

$$\int \sin.^p x \cos.^q x dx = -\frac{\sin.^{p-1}x\cos.^{q+1}x}{p+q} + \frac{p-1}{p+q}\int \sin.^{p-2}x\cos.^q x dx \qquad (1);$$

and from (B′),

$$\int \sin.^p x \cos.^q x dx = \frac{\sin.^{p+1}x\cos.^{q-1}x}{p+q} + \frac{q-1}{p+q}\int \sin.^p x \cos.^{q-2} x dx. \qquad (2).$$

When q is positive, by the application of (2),

$$\int \sin.^p x \cos.^q x dx$$

will finally depend on $\int \sin.^p x dx$, or on $\int \sin.^p x \cos. x dx$, according as q is even or odd.

By making $q=0$, in (1), we have

$$\int \sin.^p x dx = -\frac{\sin.^{p-1} x \cos. x}{p} + \frac{p-1}{p}\int \sin.^{p-2} x dx;$$

and thus, if p be a positive integer, and even, $\int \sin.^p x dx$ will at last depend on $\int dx = x$; and if p be a positive integer, and odd, the final integral to be found is $\int \sin. x dx = -\cos. x$. In the second case, that is, when q is odd, we have

$$\int \sin.^p x \cos. x dx = \int \sin.^p x d \sin. x = \frac{\sin.^{p+1} x}{p+1}.$$

It is therefore always possible to find the integral

$$\int \sin.^p x \cos.^q x dx$$

when p and q are entire and positive.

Formulæ 1 and 2 are inapplicable when $p=-q$; but in this case

$$\begin{aligned}\int \sin.^p x \cos.^q x dx = \int \tan.^p x dx &= \int \tan.^{p-2} x \tan.^2 x dx \\ &= \int \tan.^{p-2} x (\sec.^2 x - 1) dx \\ &= \int \tan.^{p-2} x d \tan. x - \int \tan.^{p-2} x dx \\ &= \frac{\tan.^{p-1} x}{p-1} - \int \tan.^{p-2} x dx. \quad (3).\end{aligned}$$

This formula serves for the reduction of the exponent of $\tan. x$; and the integration will at last depend on that of

$$\int dx = x, \text{ or } \int \tan. x dx = l \cos. x,$$

according as p is even or odd.

221. In (1) of the preceding article, making $q=0$, we have

$$\int \sin.^p x dx = -\frac{\sin.^{p-1} x \cos. x}{p} + \frac{p-1}{p}\int \sin.^{p-2} x dx;$$

and hence, when p is even,

$$\int \sin.^p x dx = -\frac{\cos. x}{p}\left\{\sin.^{p-1}x + \frac{p-1}{p-2}\sin.^{p-3}x + \frac{(p-1)(p-3)}{(p-2)(p-4)}\sin.^{p-5}x + \cdots + \frac{(p-1)(p-3)\ldots 3.1}{(p-2)(p-4)\ldots 4.2}\sin. x + \frac{(p-1)(p-3)\ldots 3.1}{(p-2)(p-4)\ldots 4.2}\frac{x}{p}\right\} \quad (1);$$

and, when p is odd,

$$\int \sin.^p x dx = -\frac{\cos. x}{p}\left\{\sin.^{p-1}x + \frac{p-1}{p-2}\sin.^{p-3}x + \cdots + \frac{(p-1)(p-3)\ldots 2}{(p-2)(p-4)\ldots 1}\right\} \quad (2).$$

In like manner, by making $p = 0$ in (2) of the preceding article, we should have formulæ for

$$\int \cos.^q x dx.$$

EXAMPLES.

1. $$\int \frac{x^n dx}{(2ax - x^2)^{\frac{1}{2}}} = -\frac{x^{n-1}(2ax - x^2)^{\frac{1}{2}}}{n} + \frac{2n-1}{n}a\int \frac{x^{n-1}dx}{(2ax-x^2)^{\frac{1}{2}}}.$$

This will finally lead to

$$\int \frac{dx}{(2ax - x^2)^{\frac{1}{2}}} = \text{ver. sin.}^{-1}\frac{x}{a}.$$

2. $$\int \frac{x^n dx}{(2ax + x^2)^{\frac{1}{2}}} = \frac{x^{n-1}(2ax + x^2)^{\frac{1}{2}}}{n} - \frac{2n-1}{n}a\int \frac{x^{n-1}dx}{(2ax+x^2)^{\frac{1}{2}}};$$

and ultimately we should have (Ex. 7, p. 328)

$$\int \frac{dx}{(2ax+x^2)^{\frac{1}{2}}} = l(x+a+\sqrt{2ax+x^2}).$$

3. $$\int (a^2-x^2)^n dx = \frac{x(a^2-x^2)^n}{2n+1} + \frac{2n}{2n+1} a^2 \int (a^2-x^2)^{n-1} dx.$$

If n be a fraction belonging to the series $\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots$, this process of reduction would at last lead to

$$\int \frac{dx}{(a^2-x^2)^{\frac{1}{2}}} = \sin.^{-1}\frac{x}{a}.$$

4. $$\int (a^2-x^2)^{\frac{3}{2}} dx = \frac{x}{4}(a^2-x^2)^{\frac{3}{2}} + \frac{3}{8}a^2 x(a^2-x^2)^{\frac{1}{2}} + \frac{3a^4}{8}\sin.^{-1}\frac{x}{a}.$$

5. $$\int (a^2+x^2)^n dx = \frac{x(a^2+x^2)^n}{2n+1} - \frac{2n}{2n+1} a^2 \int (a^2+x^2)^{n-1} dx.$$

When n is any one of the fractions $\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots$, the integral will finally depend on

$$\int \frac{dx}{(a^2+x^2)^{\frac{1}{2}}} = l\left\{x+(a^2+x^2)^{\frac{1}{2}}\right\}.$$

6. $$\int \frac{dx}{(a^2-x^2)^n} = \frac{1}{2(n-1)a^2}\,\frac{x}{(a^2-x^2)^{n-1}} + \frac{2n-3}{2(n-1)}\,\frac{1}{a^2}\int \frac{dx}{(a^2-x^2)^{n-1}}.$$

When n is one of the series of fractions $\frac{3}{2}, \frac{5}{2}, \frac{7}{2} \ldots$, the integration will finally depend on that of

$$\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{x}{a^2(a^2 - x^2)^{\frac{1}{2}}},$$

by Formula D.

7. $$\int \frac{dx}{(a^2 - x^2)^{\frac{5}{2}}} = \frac{1}{3a^2} \frac{x}{(a^2 - x^2)^{\frac{3}{2}}} + \frac{2}{3a^2} \frac{x}{a^2(a^2 - x^2)^{\frac{1}{2}}}.$$

8. $$\int \frac{dx}{x^n(a^2 - x^2)^{\frac{1}{2}}} = -\frac{(a^2 - x^2)^{\frac{1}{2}}}{(n-1)a^2x^{n-1}}$$
$$+ \frac{n-2}{n-1} \frac{1}{a^2} \int \frac{dx}{x^{n-2}(a^2 - x^2)^{\frac{1}{2}}}.$$

When n is even, the application of the formula will lead to

$$\int \frac{dx}{x^2(a^2 - x^2)^{\frac{1}{2}}} = -\frac{(a^2 - x^2)^{\frac{1}{2}}}{a^2x};$$

and, when n is odd, the integral will depend on that of

$$\int \frac{dx}{x(a^2 - x^2)^{\frac{1}{2}}} = \frac{1}{a} l \frac{x}{a + (a^2 - x^2)^{\frac{1}{2}}} = \frac{1}{a} l \frac{a - (a^2 - x^2)^{\frac{1}{2}}}{x}$$

(Ex. 17, p. 332).

9. $$\int \frac{x^n dx}{(a + bx)^{\frac{1}{3}}} = \frac{3x^n(a + bx)^{\frac{2}{3}}}{(3n + 2)b} - \frac{3na}{3n + 2b} \int \frac{x^{n-1}dx}{(a + bx)^{\frac{1}{3}}}.$$

By the application of this formula, the exponent n will at last be reduced to zero, and the integration will depend on that of

$$\int \frac{dx}{(a + bx)^{\frac{1}{3}}} = \frac{3}{2b} (a + bx)^{\frac{2}{3}}.$$

10. $$\int \frac{x^2 dx}{(a + bx)^{\frac{1}{3}}} = (a + bx)^{\frac{2}{3}} \left\{ \frac{3x^2}{8b} - \frac{9ax}{20b^2} + \frac{27a^2}{40b^3} \right\}.$$

11. $$\int \frac{x^n dx}{(a+b+cx^2)^{\frac{1}{2}}} = \frac{x^{n-1}(a+bx+cx^2)^{\frac{1}{2}}}{nc}$$

$$-\frac{n-1}{n}\,\frac{a}{c}\int \frac{x^{n-2}dx}{(a+bx+cx^2)^{\frac{1}{2}}}$$

$$-\frac{2n-1}{2n}\,\frac{b}{c}\int \frac{x^{n-1}dx}{(a+bx+cx^2)^{\frac{1}{2}}};$$

and, ultimately, we shall have to find

$$\int \frac{xdx}{(a+bx+cx^2)^{\frac{1}{2}}} = \frac{(a+bx+cx^2)^{\frac{1}{2}}}{c} - \frac{b}{2c}\int \frac{dx}{(a+bx+cx^2)^{\frac{1}{2}}}:$$

but the integration of $\dfrac{dx}{(a+bx+cx^2)^{\frac{1}{2}}}$ has been explained (Ex. 7, p. 328).

12. $$\int \frac{x^2 dx}{(2-2x+x^2)^{\frac{1}{2}}} = \frac{x+3}{2}(2-2x+x^2)^{\frac{1}{2}}$$

$$+\frac{1}{2}\,l\left\{x-1+(2-2x+x^2)^{\frac{1}{2}}\right\}.$$

13. $$\int x(2ax-x^2)^{\frac{1}{2}}dx = -\frac{1}{3}(2ax-x^2)^{\frac{3}{2}}$$

$$+a\int(2ax-x^2)^{\frac{1}{2}}dx.$$

14. $$\int_0^{2a} x(2ax-x^2)^{\frac{1}{2}}dx = \frac{\pi a^3}{2}.$$

15. $$\int x^2(2ax-x^2)^{\frac{1}{2}}dx = -\frac{x}{4}(2ax-x^2)^{\frac{3}{2}}$$

$$+\frac{5a}{4}\int x(2ax-x^2)^{\frac{1}{2}}dx.$$

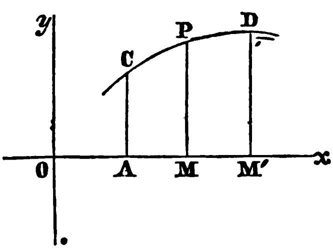
16. $$\int_0^{2a} x^2(2ax-x^2)^{\frac{1}{2}}dx = \frac{5a^4\pi}{8}.$$

17. $$\int \frac{\sin.^2 x dx}{\cos.^3 x} = \frac{\sin. x}{2\cos.^2 x} + \frac{1}{4}\,l\,\frac{1-\sin. x}{1+\sin. x}.$$

SECTION IV.

GEOMETRICAL SIGNIFICATION AND PROPERTIES OF DEFINITE INTEGRALS. — ANOTHER DEMONSTRATION OF TAYLOR'S THEOREM. — DEFINITE INTEGRALS IN WHICH ONE OF THE LIMITS BECOMES INFINITE. — DEFINITE INTEGRALS IN WHICH THE FUNCTION UNDER THE SIGN $\int$ BECOMES INFINITE. — DEFINITE INTEGRALS THAT BECOME INDETERMINATE. — INTEGRATION BY SERIES.

222. ASSUME CPD to be the curve of which the equation, when referred to the rectangular axes Ox, Oy, is $y = f(x)$. It has been shown (Art. 164) that $f(x)dx$ is the differential of the area of a segment of the curve terminated by a variable ordinate; and therefore $\int f(x)dx$ is to be regarded as the expression for the area bounded by the curve, the axis of abscissæ, and any two ordinates whatever. If this integral be taken between assigned values, a and b, for x, the area will be limited in the direction of the axis of x by the ordinates corresponding to these values of x. But the arbitrary constant may be determined by the condition that the area shall be nothing for $x = OA = a$; that is, shall be limited on one side by the fixed ordinate AC, while on the other side it is bounded by a variable ordinate corresponding to the variable abscissa $OM = x$.



If $\int f(x)dx = \psi(x) + C$, then, by the above condition, we should have

$$\psi(a) + C = 0,\ \ C = -\psi(a);$$

and

$$\int f(x)dx = \psi(x) - \psi(a)$$

is the expression for an indefinite area taking its origin from the fixed ordinate AC.

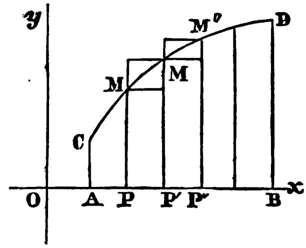
When the other value of x, $x = OM' = b$, is assigned, the integral becomes definite, and we have

$$\int_a^b f(x)dx = \psi(b) - \psi(a).$$

Definite integrals, when applied in the determination of the length of curves, the surfaces and volumes of solids, admit of a like interpretation.

223. In Art. 192, it was shown that a definite integral was to be regarded as the limit of the sum of an infinite number of very small terms.

To illustrate this proposition geometrically, let $y = f(x)$ be the equation of the curve CMD referred to rectangular co-ordinate axes, and suppose that, between the assumed limits $x = a$, $x = b$, y increases continuously.



Then $f(x)\Delta x$ measures the area of one of the small rectangles MP', $M'P''$...; and if $\Sigma f(x)\Delta x$ denotes the sum of all these rectangles, the area $ACDB$ included between the curve, the axis of x, and the ordinates $y = f(a)$, $y = f(b)$, will be expressed by

$$\lim. \Sigma f(x)\Delta x = \int_a^b f(x)dx = \psi(b) - \psi(a),$$

if $\psi(x)$ be the function of which $f(x)$ is the differential co-efficient.

224. The order in which the limits of a definite integral are taken may be inverted, provided the sign of the result be changed for

$$\int_a^b f(x)dx = \psi(b) - \psi(a),$$

$$\int_b^a f(x)dx = \psi(a) - \psi(b):$$

$$\therefore \qquad \int_b^a f(x)dx = -\int_a^b f(x)dx.$$

Also, if c be a value of x intermediate to the limits a and b, we have

$$\int_a^c f(x)dx = \psi(c) - \psi(a),$$

$$\int_c^b f(x)dx = \psi(b) - \psi(c),$$

$$\int_a^b f(x)dx = \psi(b) - \psi(a);$$

$$\therefore \qquad \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx:$$

and generally, if there be any number of values c, c', c''..., between the values a and b, it may be shown that

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^{c'} f(x)dx \cdots + \int_{c^{(n)}}^b f(x)dx.$$

225. Let $f(x)$, $\varphi(x)$, be two functions of x, so related that $f(x) > \varphi(x)$ for all values of x from $x = a$ to $x = b$; then, taking $f(x) - \varphi(x)$ for the differential co-efficient of another function, we should have

$$\int_a^b \Big\{ f(x) - \varphi(x) \Big\} dx > 0;$$

since the derivative $f(x) - \varphi(x)$ is constantly positive between the limits a and b, and the function $\int \left\{ f(x) - \varphi(x) \right\} dx$ is an increasing function of x: hence

$$\int_a^b f(x)dx > \int_a^b \varphi(x)dx.$$

Also if $\varphi_1(x)$ is another function of x, such that $\varphi_1(x) > f(x)$, for all values of x between the limits a and b, we should have

$$\int_a^b f(x)dx < \int_a^b \varphi_1(x)dx:$$

therefore

$$\int_a^b \varphi_1(x)dx > \int_a^b f(x)dx > \int_a^b \varphi(x)dx.$$

When a given differential cannot be integrated, it is desirable, and sometimes possible, to find two other integrals between which the required integral, at assigned limits, will be included.

EXAMPLE. $\int_0^{\frac{1}{2}} \frac{dx}{(1-x^3)^{\frac{1}{2}}}$. For values of x between 0 and 1, we have

$$1 < \frac{1}{(1-x^3)^{\frac{1}{2}}} < \frac{1}{(1-x^2)^{\frac{1}{2}}}:$$

$$\therefore \quad \int_0^{\frac{1}{2}} dx < \int_0^{\frac{1}{2}} \frac{dx}{(1-x^3)^{\frac{1}{2}}} < \int_0^{\frac{1}{2}} \frac{dx}{(1-x^2)^{\frac{1}{2}}}:$$

$$\therefore \quad 0.5 < \int_0^{\frac{1}{2}} \frac{dx}{(1-x^3)^{\frac{1}{2}}} < \sin.^{-1}\frac{1}{2} = 0.5236.$$

226. Demonstration of Taylor's Theorem dependent on the properties of definite integrals.

The equation

$$f(x+h) - f(x) = \int_0^h f'(x+h-t)dt$$

is identically true; but successive integration by parts gives

$$\int_0^t f'(x+h-t)dt = tf'(x+h-t) + \int_0^t tf''(x+h-t)dt,$$

$$\int_0^t tf''(x+h-t)dt = \frac{t^2}{1.2}f''(x+h-t) + \int_0^t \frac{t^2}{1.2}f'''(x+h-t)dt,$$

$$\int_0^t \frac{t^2}{1.2}f'''(x+h-t)dt = \frac{t^3}{1.2.3}f'''(x+h-t) + \int_0^t \frac{t^3}{1.2.3}f''''(x+h-t)dt,$$

. . . .

$$\int_0^t \frac{t^{n-1}}{1.2\ldots(n-1)}f^{(n)}(x+h-t)dt = \frac{t^n}{1.2\ldots n}f^{(n)}(x+h-t) + \int_0^t \frac{t^n}{1.2\ldots n}f^{(n+1)}(x+h-t)dt.$$

Making $t=h$ in these equations, and then adding them member to member, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{1.2}f''(x) + \cdots + \frac{h^n}{1.2\ldots n}f^{(n)}(x) + \frac{h^n}{1.2..n}\int_0^h f^{(n+1)}(x+h-t)dt.$$

If the function to be expanded, and also its differential coefficients up to the order denoted by $n+1$, are finite, and continuous between the limits x and $x+h$, the residual term $\frac{h^n}{1.2..n}\int_0^h f^{(n+1)}(x+h-t)dt$ may be replaced by

$$\frac{h^n}{1.2..n+1}f^{(n+1)}(x+\theta h) = R,$$

and the expansion then agrees with that of Art. 61.

227. In what precedes, it has been supposed that the limits a and b of the definite integral $\int_a^b f(x)dx$ were finite, and that the function $f(x)$ was also finite, and continuous between these same limits. It may happen that one of the limits, b, becomes infinite while the other is finite, the function remaining finite and continuous. Then the value of the integral is the limit of the value of $\int_a^b f(x)dx$ when b is increased without limit. This value may be finite, infinite, or indeterminate.

EXAMPLE 1. $\int_0^\infty e^{-x}dx.$

For the indefinite integral, we have

$$\int e^{-x}dx = -e^{-x} + C:$$

$$\therefore \qquad \int_0^b e^{-x}dx = 1 - \frac{1}{e^b},$$

$$\int_0^\infty e^{-x}dx = 1 - \frac{1}{e^\infty} = 1.$$

Ex. 2. $\int_0^\infty e^x dx.$

The indefinite integral is

$$\int e^x dx = e^x + C:$$

$$\therefore \qquad \int_0^\infty e^x dx = e^\infty - 1 = \infty.$$

Ex. 3. $\int_0^\infty \frac{dx}{x^2 + a^2}.$

We have $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan.^{-1}\frac{x}{a} + C:$$

$$\therefore \qquad \int_0^\infty \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan.^{-1}\infty = \frac{1}{a}\frac{\pi}{2}.$$

Ex. 4. $\int_0^\infty \cos.x\,dx$.

In this case, $\int \cos.x\,dx = \sin.x + C$; and, taking the integral between 0 and the finite limit b, we have

$$\int_0^b \cos.x\,dx = \sin.b;$$

but, when b becomes infinite, the value of $\sin.b$ will be indeterminate, though confined within the limits 0 and 1.

The following investigation will sometimes enable us to decide whether the definite integral $\int_a^b f(x)dx$ is finite or infinite for $b = \infty$ or $b = -\infty$.

First suppose that b is very great, but not infinite, and let c be a number comprised between a and b; then (Art. 224)

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Since $f(x)$ is finite, $\int_a^c f(x)dx$ is also finite; and it remains only to examine the value of $\int_c^b f(x)dx$ when b becomes infinite.

Put $f(x) = \frac{\varphi(x)}{x^n}$, $\varphi(x)$ being a function that remains finite for all values of x greater than c. If A denotes the greatest and B the least of the values of $\varphi(x)$ for all values of x greater than c, we shall have

$$\frac{A}{x^n} > f(x) > \frac{B}{x^n}:$$

$$\therefore \qquad \int_c^b f(x)dx < A\int_c^b \frac{dx}{x^n},$$

or $$\int_c^b f(x)dx < \frac{A}{n-1}\left(\frac{1}{c^{n-1}} - \frac{1}{b^{n-1}}\right).$$

Now, when $n > 1$, the second member of this last inequality for $b = \infty$ reduces to $\frac{A}{n-1}\frac{1}{c^{n-1}}$: hence, in this case, we know that the integral $\int_a^\infty f(x)dx$ has a finite value. When $n < 1$, we have

$$\int_c^b f(x)dx > B\int_c^b \frac{dx}{x^n}:$$

$$\therefore \quad \int_c^b f(x)dx > \frac{B}{1-n}(b^{1-n} - c^{1-n}).$$

Now, when $1 - n > 0$, the second member of this inequality becomes infinite for $b = \infty$: hence $\int_c^b f(x)dx$, and therefore $\int_a^b f(x)dx$ is infinite for $b = \infty$.

If $n = 1$, then

$$\int_c^b f(x)dx > B\int_c^b \frac{dx}{x} = Bl\left(\frac{b}{c}\right);$$

but $l\left(\frac{b}{c}\right) = \infty$ when $b = \infty$: hence $\int_a^\infty f(x)dx = \infty$.

Putting $f(x)$ under the form $\frac{\varphi(x)}{x^n}$, $\varphi(x)$ being a function that is finite for all values of x between $-\infty$ and some value less than b, it may be shown in like manner that $\int_{-\infty}^b f(x)dx$ is finite if $n > 1$, and infinite if $n < 1$ or $n = 1$.

Thus, if it be possible to put $f(x)$ under the form $\frac{\varphi(x)}{x^n}$, and the condition imposed on $\varphi(x)$ be satisfied, we can decide whether the integral $\int_a^b f(x)dx$ is finite or infinite when one of its limits becomes $+\infty$ or $-\infty$.

228. Definite integrals in which the function under the sign of integration becomes infinite between or at the limits.

The function $f(x)$ may become infinite at one of the limits, b, of the integral $\int_a^b f(x)dx$; in which case the integral is defined as the limit of $\int_a^{b-\alpha} f(x)dx$ when α is decreased without limit. In like manner, if $f(x)$ becomes infinite for $x = a$, then $\int_a^b f(x)dx$ is the limit of $\int_{a+\alpha}^b f(x)dx$ when α is indefinitely decreased. Finally, if $f(c) = \infty$, c being comprised between a and b, we should have

$$\int_a^b f(x)dx = \lim. \int_a^{c-\alpha} f(x)dx + \lim. \int_{c+\beta}^b f(x)dx,$$

when α and β are decreased without limit. Should there be more than one value of x for which $f(x)$ becomes infinite between the limits a and b, we learn from what precedes how to define the integral $\int_a^b f(x)dx$.

229. It may sometimes be decided whether the integral $\int_a^b f(x)dx$ is finite or infinite when $f(x)$ is infinite at one of the limits. Suppose $f(b) = \infty$, and let $f(x) = \dfrac{\varphi(x)}{(b-x)^n}$; $\varphi(x)$ being finite for $x = b$ and for all values of $x < b$, and n being > 0.

If c be a number comprised between a and b, we have

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Now $\int_a^c f(x)dx$ is finite; and hence $\int_a^b f(x)dx$ will be finite or infinite according as $\int_c^b f(x)dx$ is finite or infinite.

Denote by A the greatest and by B the least of the values of $\varphi(x)$ for values of x included between c and b. If $n < 1$,

we shall have, for such values of x, $f(x) < \frac{A}{(b-x)^n}$; and therefore

$$\int_c^{b-\alpha} f(x)dx < \int_c^{b-\alpha} \frac{Adx}{(b-x)^n} = \frac{A}{1-n}\left\{(b-c)^{1-n} - \alpha^{1-n}\right\}.$$

When α converges towards 0, the second member of this inequality converges towards the finite value $\frac{A}{1-n}(b-c)^{1-n}$: hence, in this case, the value of lim. $\int_c^{b-\alpha} f(x)dx$, and therefore of $\int_a^b f(x)dx$, is finite.

But if $n > 1$, the proposed integral is infinite; for, since $f(x) > \frac{B}{(b-x)^n}$, we have

$$\int_c^{b-\alpha} f(x)dx > \int_c^{b-\alpha} \frac{Bdx}{(b-x)^n} = \frac{B}{n-1}\left\{\frac{1}{\alpha^{n-1}} - \frac{1}{(b-c)^{n-1}}\right\};$$

and it is evident, that, when α becomes 0, the second member of this inequality becomes infinite: hence, under the supposition, $\int_a^b f(x)dx$ is infinite.

In like manner, when $n = 1$, we have $f(x) > \frac{B}{b-x}$; and therefore

$$\int_a^{b-\alpha} f(x)dx > \int_c^{b-\alpha} \frac{Bdx}{b-x} = Bl\frac{b-c}{\alpha}.$$

But $Bl\frac{b-c}{\alpha}$ becomes infinite when α vanishes: hence $\int_c^b f(x)dx$, and therefore $\int_a^b f(x)dx$, is then infinite.

EXAMPLE 1. $\int_a^b \frac{Pdx}{\sqrt{2b^2 - bx - x^2}}$.

P being a function of x that remains finite for all finite

values of x, and a and b being two positive quantities, we have

$$\frac{P}{\sqrt{2b^2 - bx - x^2}} = \frac{P}{\sqrt{2b + x}}\,\frac{1}{\sqrt{b - x}} = \frac{\varphi(x)}{(b - x)^{\frac{1}{2}}}$$

by putting $\frac{P}{\sqrt{2b + x}} = \varphi(x)$.

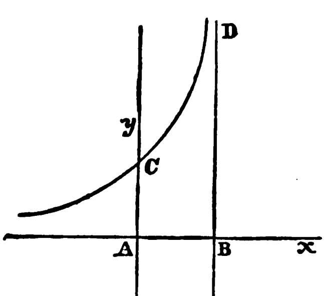
Since the exponent of $b - x$ is less than 1, it follows from the rule just established that the proposed integral has a finite value.

Ex. 2. $\int_0^1 \frac{dx}{\sqrt{1 - x}}$. We have $\int \frac{dx}{\sqrt{1 - x}} = -2\sqrt{1 - x}$:

$$\therefore \quad \int_0^{1-\alpha} \frac{dx}{\sqrt{1 - x}} = 2 - 2\sqrt{\alpha}: \quad \therefore \int_0^1 \frac{dx}{\sqrt{1 - x}} = 2.$$

The expression $\frac{dx}{\sqrt{1 - x}}$ is the differential of the area included between the axis of x, and the curve having $y = \frac{1}{\sqrt{1 - x}}$ for its equation.

This curve has two asymptotes; the one the axis of x, and the other a parallel to the axis of y, and at the distance $+1$ from it. It is seen from the figure the $\int_0^1 \frac{dx}{\sqrt{1 - x}}$ represents the area bounded by AC, AB, the curve, and its asymptote BD; and this area, although it extends indefinitely in the direction of the asymptote BD, still has the finite value 2.



230. A definite integral may become indeterminate, as is the case for

$$\int_0^\infty \sin. x dx = \cos. \infty - \cos. 0,$$

since $\cos. x$, when x is indefinitely increased, does not converge towards any determinate limit.

For another example, take $\int_{-a}^{+b} \frac{dx}{x}$, in which a and b are any two positive quantities whatever. Since $\frac{1}{x}$ becomes infinite for the value $x = 0$, which is comprised between $-a$ and $+b$, we put

$$\int_{-a}^{+b} \frac{dx}{x} = \lim. \int_{-a}^{-\alpha} \frac{dx}{x} + \lim. \int_{+\beta}^{+b} \frac{dx}{x};$$

α and β being numbers numerically less than a and b respectively, and the limits indicated being those answering to $\alpha = 0$, $\beta = 0$. But

$$\int_{-a}^{-\alpha} \frac{dx}{x} = l\alpha - la, \int_{\beta}^{b} \frac{dx}{x} = lb - l\beta:$$

$$\therefore \quad \int_{-a}^{-\alpha} \frac{dx}{x} + \int_{\beta}^{b} \frac{dx}{x} = lb - l\beta + l\alpha - la = l\left(\frac{b}{a}\right) + l\left(\frac{\alpha}{\beta}\right).$$

Therefore $$\int_{-a}^{+b} \frac{dx}{x} = l\left(\frac{b}{a}\right) + \lim. l\left(\frac{\alpha}{\beta}\right).$$

The first term $l\left(\frac{b}{a}\right)$ in the value of this integral is determinate; but, since the variables α and β are entirely independent of each other, the term $\lim. l\left(\frac{\alpha}{\beta}\right)$ does not converge towards any fixed limit, and the integral is therefore indeterminate.

231. When the integral of Xdx is required, and X can be developed into a converging series,

$$X = u_1 + u_2 + u_3 + \cdots + u_n + r_n \quad (1),$$

we shall have, after multiplying by dx, and integrating between the limits a and b,

$$\int_a^b Xdx = \int_a^b u_1 dx + \int_a^b u_2 dx + \cdots + \int_a^b u_n dx + \int_a^b r_n dx \quad (2).$$

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If series (1) is converging for $x = a$, $x = b$, and also for all values of x between a and b, we may assume $r_n < \alpha$; α being less than any assigned quantity when n is taken sufficiently great. Whence

$$\int_a^b r_n dx < \int_a^b \alpha dx, \text{ or} \int_a^b r_n dx < \alpha(b-a).$$

Therefore $\int_a^b r_n dx$ will decrease without limit when n is increased without limit; whence the series

$$\int_a^b u_1 dx + \int_a^b u_2 dx + \cdots + \int_a^b u_n dx$$

is converging, and its sum is the expression for $\int_a^b X dx$. The fixed limit b may be replaced by the variable x, provided no values of x are admitted which fail to render series (1) converging. We should thus have

$$\int_a^x X dx = \int_a^x u_1 dx + \int_a^x u_2 dx + \cdots + \int_a^x u_n dx \quad (3).$$

232. Formula 3 of the last article still holds true for $x = b$, even though the series $u_1 + u_2 + u_3 + \cdots$, which is supposed converging for $x < b$, becomes diverging for $x = b$, if, at the same time, Series 2 is converging.

For, however small the quantity α may be, we have

$$\int_a^{b-\alpha} X dx = \int_a^{b-\alpha} u_1 dx + \int_a^{b-\alpha} u_2 dx + \cdots + \int_a^{b-\alpha} u_n dx.$$

The two members of this equation are continuous functions of x, and are constantly equal; hence their limits for $\alpha = 0$ must be equal, and therefore

$$\int_a^b X dx = \int_a^b u_1 dx + \int_a^b u_2 dx + \cdots + \int_a^b u_n dx.$$

If the series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{1.2}f''(0) + \cdots,$$

to which the development of $f(x)$ by Maclaurin's Formula gives rise, is converging, we shall have

$$\int f(x)dx = C + xf(0) + \frac{x^2}{1.2}f'(0) + \frac{x^3}{1.2.3}f''(0) + \cdots;$$

and, if it is wished that this integral should begin with $x = 0$, C must be zero; and we then have

$$\int_0^x f(x)dx = xf(0) + \frac{x^2}{1.2}f'(x) + \frac{x^3}{1.2.3}f''(0) + \cdots$$

EXAMPLE 1. $\int \frac{dx}{1+x} = l(1+x).$

By division, or by the Binomial Formula, we have

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \pm x^{n-1} \mp \frac{x^n}{1+x}:$$

$$\therefore \quad l(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots \pm \frac{x^n}{n} \mp \int_0^x \frac{x^n dx}{1+x}.$$

When x is numerically less than 1, positive or negative, the series $1 - x + x^2 - x^3 \ldots$ is converging, and therefore so also is the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$ between the same limits for x: hence, when $x \substack{> -1 \\ < +1}$,

$$l(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$$

It may be shown by direct demonstration, that $\int_0^x \frac{x^n dx}{1+x}$ converges towards 0 as n approaches ∞.

For, if x is positive, we have $\frac{x^n}{1+x} < x^n$: therefore

$$\int_0^x \frac{x^n dx}{1+x} < \int_0^x x^n dx = \frac{x^{n+1}}{n+1}.$$

Now, as n increases, $\frac{x^{n+1}}{n+1}$ approaches 0; and consequently, in stopping with any term of the series, the error will be less than the following term, and will be additive or subtractive according as the last term taken is of an even or odd order.

If x is negative, and α denotes a number greater than x, but less than 1, we have $\frac{x^n}{1-x} < \frac{x^n}{1-\alpha}$; and therefore

$$\int_0^x \frac{x^n dx}{1-x} < \frac{x^{n+1}}{(n+1)(1-\alpha)}$$

The limit of the second member of this inequality for $n=\infty$ is zero. In this case, the error is always numerically additive.

When $x=1$, the series $1-x+x^2-x^3+\cdots$ is no longer converging; but the series $x-\frac{x^2}{1}+\frac{x^3}{3}-\cdots$ is (Art. 231), and will represent the value of $l2$: hence

$$l2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

Ex. 2. $\int_0^x \frac{dx}{1+x^2} = \tan.^{-1} x.$

We have $\frac{1}{1+x^2} = 1-x^2+x^4-x^6\cdots \pm x^{n-1} \mp \frac{x^{n+1}}{1+x^2}$, n being an odd number, and positive. Integrating, and taking for $\tan.^{-1}x$ the least positive arc having x for the tangent, we find

$$\tan.^{-1} x = x - \frac{x^3}{3}+\frac{x^5}{5}\cdots \pm \frac{x^n}{n} \mp \int_0^x \frac{x^{n+1}dx}{1+x^2}.$$

The series $1-x^2+x^4-x^6\ldots$ ceases to be converging for $x=1$; but the series $x-\frac{x^3}{3}+\frac{x^5}{5}\cdots$ is still converging for this value of x: hence

$$\tan.^{-1} x = \frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\cdots$$

Ex. 3. $\int_0^x \frac{dx}{\sqrt{1-x^2}} = \sin.^{-1} x.$

We have

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1}{2}\frac{3}{4}x^4 + \frac{1}{2}\frac{3}{4}\frac{5}{6}x^6 + \cdots \quad (1).$$

From this, by multiplying by dx, and integrating, we find

$$\sin.^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7}\cdots,$$

a converging series when $x \gtrless {}^{-1}_{+1}$; since Series 1 is converging between these limits.

The series

$$1 + \frac{1}{2}x^2 + \frac{1}{2}\frac{3}{4}x^4 + \frac{1}{2}\frac{3}{4}\frac{5}{6}x^6$$

is not converging when $x = 1$; but since, for $x = 1 = \sin.$ the series

$$x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7}$$

is converging (Art. 231),* we have

$$\frac{\pi}{2} = 1 + \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{3}{4}\frac{1}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{7} + \cdots$$

A still more converging series is found by making

$$x = \frac{1}{2} = \sin.\frac{\pi}{6}:$$

whence

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\frac{1}{2^3.3} + \frac{1}{2}\frac{3}{4}\frac{1}{2^5.5} + \cdots$$

* Space does not allow the proof of convergence or divergence when these conditions are asserted relative to the series involved in the last three examples. (See Art. 68.)

233. By integrating $f(x)dx$ by parts, we have

$$\int f(x)dx = xf(x) - \int xf'(x)dx,$$

$$\int xf'(x)dx = \frac{x^2}{2} f'(x) - \int \frac{x^2}{2} f''(x)dx,$$

$$\int xf''(x)dx = \frac{x^3}{3} f''(x) - \int \frac{x^3}{3} f'''(x)dx.$$

. . . .

The combination of these results gives

$$\int f(x)dx = xf(x) - \frac{x^2}{1.2} f'(x) + \frac{x^3}{1.2.3} f''(x) + \cdots$$

$$+ \frac{(-1)^{n-1} x^{n-1}}{1.2..n} f^{(n-1)}(x) + \frac{(-1)^n x^n}{1.2..n} \int x^n f^{(n)}(x)\,dx.$$

This is the series of John Bernouilli, and may be advantageously used in many cases: for example, if $f(x)$ be a rational algebraic function of $(n-1)^{\text{th}}$ degree, $f^{(n)}(x)$ is 0, and the series will terminate; or there may be cases when

$$\int x^n f^{(n)}(x)dx$$

can be more readily found than $\int f(x)dx$, or when only an approximate value of $\int_0^a f(x)dx$ may be required, and the integral $\int_0^a x^n f^{(n)}(x)dx$ may be small enough to be neglected without sensible error.

234. Assuming $\int f(x)dx = \varphi(x)$, and making $x = x + h$, we have, by Taylor's Formula,

$$\varphi(x+h) - \varphi(x) = h\varphi'(x) + \frac{h^2}{1.2}\varphi''(x) + \cdots \quad (1).$$

But, because $\int f(x)dx = \varphi(x)$, we have

$$f(x) = \varphi'(x),\ f'(x) = \varphi''(x),\ f''(x) = \varphi'''(x).$$

These values, substituted in (1), give

$$\varphi(x+h)-\varphi(x)=hf(x)+\frac{h^2}{1.2}f'(x)+\frac{h^3}{1.2.3}f''(x)+\cdots$$

In this series, making $x=a$, $h=b-a$, and denoting by $A_1, A_2, A_3 \ldots$, what $f(x), f'(x), f''(x) \ldots$, become under this supposition, then $\varphi(x+h)-\varphi(x)$ becomes

$$\varphi(b)-\varphi(a)=\int_a^b f(x)dx,$$

and we have

$$\int_a^b f(x)dx=A_1(b-a)+\frac{A_2}{1.2}(b-a)^2+\frac{A_3}{1.2.3}(b-a)^3\ldots$$

This series enables us to find the approximate value of the definite integral $\int_a^b f(x)dx$ when $b-a$ is sufficiently small to make the series converging. When this is not the case, or when the series does not converge rapidly enough for our purposes, put $b-a=n\alpha$, and take the integral successively between the limits a and $a+\alpha$, $a+\alpha$ and $a+2\alpha$, and so on, denoting the results by

$$B_1\alpha+\frac{B_2}{1.2}\alpha^2+\frac{B_3}{1.2.3}\alpha^3\ldots,$$

$$C_1\alpha+\frac{C_2}{1.2}\alpha^2+\frac{C_3}{1.2.3}\alpha^3\ldots,$$

$$D_1\alpha+\frac{D_2}{1.2}\alpha^2+\frac{D_3}{1.2.3}\alpha^3\ldots;$$

then (Art. 224) we have

$$\int_a^b f(x)dx=(B_1+C_1+D_1+\cdots)\alpha+(B_2+C_2+D_2+\cdots)\alpha^2 \\ +(B_3+C_3+D_3+\cdots)\alpha^3+\cdots,$$

a series that may be made to converge as rapidly as we please by making α sufficiently small.

SECTION V.

GEOMETRICAL APPLICATIONS.

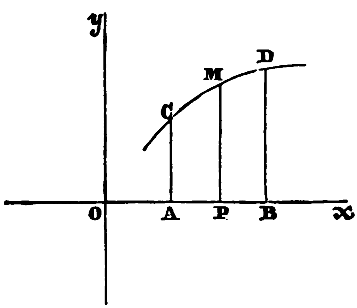
QUADRATURE OF PLANE CURVES REFERRED TO RECTILINEAR CO-ORDINATES.—QUADRATURE OF PLANE CURVES REFERRED TO POLAR CO-ORDINATES.

235. ***The quadrature of a curve*** is the operation of finding the area bounded in whole or in part by the curve.

If u denote the indefinite area limited by the curve, the axis of x, and any two ordinates, it was found (Art. 164) that

$$du = ydx = f(x)dx;$$

$y = f(x)$ being the equation of the curve referred to the rectangular axis Ox, Oy.



If it is desired to have the area limited on one side by the fixed ordinate CA, corresponding to the abscissa $x = OA = a$, the integral must begin at $x = a$; and we have

$$u = \int_a^x f(x)dx.$$

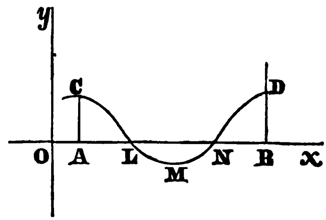
Finally, if the area is to be limited on the other side by the ordinate BD, corresponding to $x = OB = b$, we have

$$u = \text{area } ACDB = \int_a^b f(x)dx.$$

When the co-ordinate axes are oblique, making with each other the angle ω, then

$$u = \text{area } ACDB = \sin.\ \omega \int_a^b f(x)dx.$$

236. The definite integral is the limit of the sum, taken between assigned limits, of an infinite number of infinitely small areas (Art. 192). Observing that $f(x)\,dx$ is equivalent to $f(x)\,\Delta x$, if we suppose $\Delta x = dx$ to be positive, the element $f(x)\,\Delta x$ will have the sign of $f(x)$. Consequently the integral will represent the difference between the sum of the segments situated above the axis of x and the sum of the segments situated below.



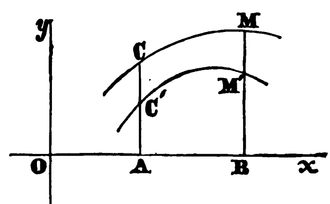
If, for example, the ordinate changes, as in the figure, from positive to negative, and then from negative to positive, the area between the ordinates AC, BD, will be

$$\int_a^h f(x)dx = ACL - LMN + NBD;$$

and if $OL = h$, $ON = k$, the sum of these segments will be expressed by

$$\int_a^h f(x)\,dx - \int_h^k f(x)dx + \int_k^b f(x)dx.$$

237. If $y = f(x)$ is the equation of the curve CM, and $y_1 = \psi(x)$ that of the curve $C'M'$, and the area bounded by these curves and the ordinates AC, BM, corresponding to $x = a$, $x = b$, is required, we have



$$\text{Area } C'CMM' = \int_a^b f(x)dx - \int_a^b \psi(x)\,dx$$

$$= \int_a^b \left\{ f(x) - \psi(x) \right\} dx.$$

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EXAMPLES.

EXAMPLE 1. The family of parabolas is represented by the equation $y^n = px^m$, m and n being positive. We have

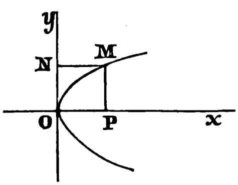
$$du = ydx = p^{\frac{1}{n}} x^{\frac{m}{n}} dx,$$

and

$$\int_0^x p^{\frac{1}{n}} x^{\frac{m}{n}} dx = \frac{n}{m+n} p^{\frac{1}{n}} x^{\frac{m+n}{n}};$$

which may be written

$$u = \frac{n}{m+n} p^{\frac{1}{n}} x^{\frac{m}{n}} x = \frac{n}{m+n} xy.$$



But xy measures the area of the rectangle $OPMN$, contained by the co-ordinates of the point M. Hence, from the above formula, we have

$$OPM : OPMN :: n : m + n,$$
$$OPM : OMN :: n : m;$$

that is, the arc of the parabola divides the rectangle constructed on the co-ordinates of its extreme point into parts having the ratio of $n : m$.

Reciprocally, the property just enunciated belongs to the parabolas alone; for the proportion

$$OPM : OMN :: n : m$$

may be written

$$u : xy - u :: n : m.$$

Hence $(m+n)u = nxy$, and, by differentiation, we have

$$(m+n)du = nxdy + nydx;$$

or, since $du = ydx$,

$$mydx = nxdy:$$

whence

$$m\frac{dx}{x} = n\frac{dy}{y}.$$

Integrating

$$nly = mlx + C, \text{ or } ly^n = lx^m + C,$$

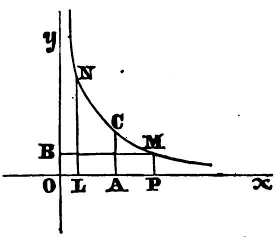
putting lp for C, we have

$$ly^n = lpx^m \text{ or } y^n = px^m$$

for the general equation of the curves which possess the property in question.

For the ordinary parabola in which $n = 2$, $m = 1$, we have $u = \frac{2}{3}xy$.

Ex. 2. The hyperbolas referred to their asymptotes are represented by the equation $x^m y^n = p$, m and n being entire and positive numbers.



Assume the asymptotes to be rectangular, and let NCM be the branch of the curve situated in the angle xOy.

Suppose $n > m$, and let $u =$ area $ACMP$, $OA = a$, $OP = x$; then

$$u = \int_a^x y dx = \int_a^x p^{\frac{1}{n}} x^{-\frac{m}{n}} dx,$$

or

$$u = \frac{n}{n-m} p^{\frac{1}{n}} \left(x^{\frac{n-m}{n}} - a^{\frac{n-m}{n}} \right).$$

As x increases, so also does u, or the area $ACMP$; and x and u become infinite at the same time. If, however, we suppose PM to be fixed, and a to decrease, the surface, while continually increasing, will remain finite; and at the limit, when $a = 0$, it reduces to $\frac{n}{n-m} p^{\frac{1}{n}} x^{\frac{n-m}{n}}$. Hence the surface $PMNL$ approaches a fixed limit as the point N approaches the asymptote Oy.

This limit, which may be written $\frac{n}{n-m}xy$, bears to the rectangle *PMBO* the constant ratio of n to $n-m$: since, denoting this limit by u, we have

$$n-m:n::xy:u=\frac{n}{n-m}xy.$$

The converse of this is also true; that is, no curves, except those represented by the equation $x^m y^n = p$, possess this property: for, from the preceding proportion, we have $u(n-m)=nxy$, which, differentiated, gives

$$(n-m)\,du = nxdy + nydx;$$

from which, by substitution and reduction, we have

$$-m\frac{dx}{x}=n\frac{dy}{y}.$$

Integrating

$$nly=-mlx+C,$$

making $C=lp$, then $ly^n = l\frac{p}{x^m}$: hence $x^m y^n = p$.

When $m=n$, the general equation takes the form $xy=p$, which is that of the equilateral hyperbola of the second degree; and we have

$$y=\frac{p}{x},\ ydx=p\frac{dx}{x},$$

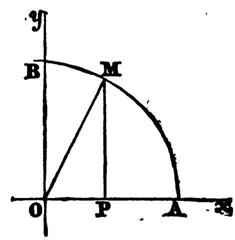
and therefore $$u=plx+C=pl\frac{x}{a}$$

by making $C=pl\frac{1}{a}$. When $p=1$ and $a=1$, we have $u=lx$; and the area is then equal to the Napierian logarithm of the abscissa.

Ex. 3. The equation of the circle, referred to its centre and rectangular axes, is

$$x^2+y^2=a^2:\ \therefore y=\sqrt{a^2-x^2};$$

and $ydx = \sqrt{a^2 - x^2}\,dx$ is the differential of the area of a segment limited by the axis of y and an arbitrary ordinate PM. Denoting this area by u, we have



$$= \int_0^x \sqrt{a^2 - x^2}\,dx.$$

Hence (Ex. 2, p. 326)

$$u = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin.^{-1} \frac{x}{a}.$$

From this we deduce the area of the sector OBM; for the area of the triangle OMP is measured by $\frac{1}{2} x \sqrt{a^2 - x^2}$, which, subtracted from the expression for u, gives

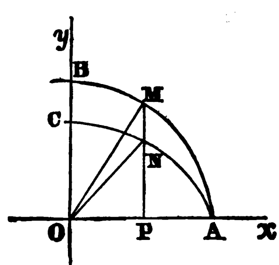
$$\text{sector } OBM = \frac{a^2}{2} \sin.^{-1} \frac{x}{a} = a\,\frac{a}{2} \sin.^{-1} \frac{x}{a} = \frac{a}{2} \text{ arc } MB:$$

that is, the area of a circular sector is measured by its arc multiplied by one-half of the radius.

Ex. 4. If a and b denote the semi-axes of an ellipse, the equation of the curve referred to its centre and axes is

$$a^2 y^2 + b^2 x^2 = a^2 b^2: \; \therefore \; y = \frac{b}{a} \sqrt{a^2 - x^2}.$$

Let u denote the area of a segment bounded by the axis of y and any ordinate, as PM; then



$$u = \frac{b}{a} \int_0^x \sqrt{a^2 - x^2}\,dx.$$

Describe a circle on $2a$ as a diameter, and denote by u' the area of the segment $BMPO$; then

$$u' = \int_0^x \sqrt{a^2 - x^2}\,dx.$$

$$\therefore \qquad u' : u :: 1 : \frac{b}{a}, \text{ or } \frac{u}{u'} = \frac{b}{a}.$$

That is, the segment of the ellipse is, to the segment of the circle which corresponds to the same abscissa, in the constant ratio of b to a; and therefore, denoting the entire area of the ellipse by A, and that of the circle by A', we have

$$A : A' :: b : a;$$

and, since $A' = \pi a^2$, it follows that

$$A = \frac{b}{a} \pi a^2 = \pi ab.$$

Hence the area of the ellipse is a mean proportional between the areas of two circles, having for diameters, the one the transverse, and the other the conjugate, axis of the ellipse.

The ordinates PM, PN, are to each other as a to b, and hence the triangles OPM, OPN, are in the same proportion; that is,

$$\frac{OPN}{OPM} = \frac{PN}{PM} = \frac{b}{a}; \text{ but } \frac{u}{u'} = \frac{b}{a}:$$

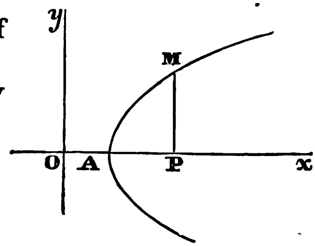
$$\therefore \qquad \frac{u - OPN}{u' - OPM} = \frac{b}{a}, \text{ or } \frac{OCN}{OBM} = \frac{b}{a};$$

and thus the area of the elliptical sector may be found in terms of the area of the corresponding circular sector.

An ellipse may be divided into any number of equal sectors when we know how to effect this division in a circle. It would only be necessary to describe a circle on the major axis of the ellipse as a diameter, then divide the circle into the required number of equal sectors, and through the points in which the circumference is divided draw ordinates to the major axis of the ellipse. The sectors formed by joining the centre with the points in which these ordinates cut the ellipse will be equal.

Ex. 5. The equation of hyperbola is $a^2y^2 - b^2x^2 = -a^2b^2$, or $y = \frac{b}{a}\sqrt{x^2 - a^2}$; and the area of the segment AMP is expressed by

$$u = \frac{b}{a}\int_0^x \sqrt{x^2 - a^2}\, dx.$$



Hence (Ex. 6, p. 328)

$$AMP = \frac{b}{2a}x\sqrt{x^2 - a^2} - \frac{ab}{2}l\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right).$$

Ex. 6. The differential equation of the cycloid (Art. 146) is

$$dx = \sqrt{\frac{y}{2r - y}}\, dy = \frac{ydy}{\sqrt{2ry - y^2}}:$$

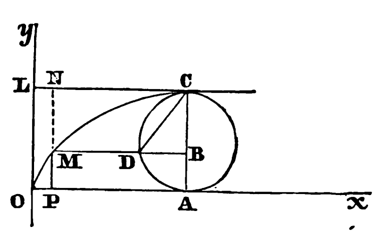
$$\therefore \qquad u = \int ydx = \int \frac{y^2 dy}{\sqrt{2ry - y^2}}.$$

This integral may be found by Ex. 1, p. 370: the following, however, is a more simple process.

Put $NM = 2r - y = z$; then, denoting the area $OLNM$ by u', we have

$$u' = \int zdx = \int (2r - y)\, dx = \int \sqrt{2ry - y^2}\, dy;$$

observing that the limits between which these integrals are taken must correspond to $z = 2r$ and $z = 2r - y$. But $\int \sqrt{2ry - y^2}\, dy$ is evidently the expression for the area of the segment of a circle of which r is the radius; the segment taking its origin at the extremity of a diameter, and having y for its base. This segment is represented by



ADB. The area $OLNM$ takes its origin from OL, and the circular segment from the point A, and both areas are zero when

$y = 0$: hence the constant of integration is 0, and we constantly have area $OLNM$ = segment ADB.

When $y = 2r$, the segment becomes the semicircle ADC, which is measured by $\frac{\pi r^2}{2}$. But

$$\text{rectangle } OLCA = OA \times AC = \pi r \times 2r = 2\pi r^2;$$

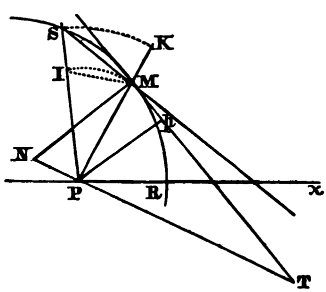
that is, the rectangle is twice the area of the generating circle: hence

$$\text{area semi-cycloid } OMCA = \frac{3}{2}\pi r^2,$$

and therefore the area bounded by a single branch of the cycloid and its base is *three times the area of the generating circle;* or, in other words, this area is *three-fourths of the rectangle having for its base the circumference, and its altitude the diameter, of the generating circle.*

Although the area of the cycloid may be said to be thus represented by a part of a rectangle, it is not a quadrable curve; for the base of the rectangle cannot be accurately determined by geometrical processes.

238. Quadrature of curves referred to polar co-ordinates.



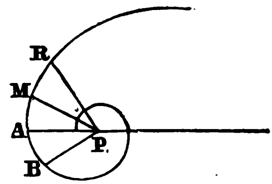
The differential of the area PRM (Art. 165) is $du = \frac{1}{2} r^2 d\theta$: hence

$$u = \frac{1}{2}\int r^2 d\theta,$$

the limits of this integral being the values of θ corresponding to the points R and M.

EXAMPLE 1. Applying this formula to the logarithmic spiral, of which the equation is $r = ae^{m\theta}$, we have

$$u = \frac{a^2}{2} \int e^{2m\theta} d\theta = \frac{a^2}{4m} e^{2m\theta} + C = \frac{r^2}{4m} + C.$$



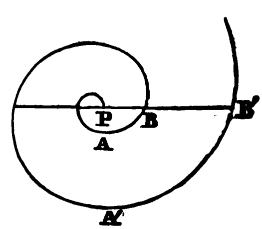
Put $PM = r'$, and in the formula make $r = r'$; then

$$0 = \frac{r'^2}{4m} + C: \therefore C = -\frac{r'^2}{4m},$$

and $u = \frac{1}{4m}\left(r^2 - r'^2\right)$.

The figure supposes PA to be the initial position of the radius vector; that is, the position at which $\theta = 0$ and $r = PA = a$, and also that θ is positive when the motion of the radius vector is in the direction of the motion of the hands of a watch. Hence, when the generating point moves in the direction from A towards B, θ is negative. Let the motion take place in this direction from the fixed radius vector $PR = r$; then, after an infinite number of revolutions, r' becomes 0, and the expression for u reduces to $u = \frac{r^2}{4m}$.

2. When the length of the radius vector of a spiral is proportional to the angle through which it has moved from its initial position, its extremity describes the spiral of Archimedes. The equation of this spiral is $r = a\theta$; and hence $r = a$ when $\theta = 57^\circ.29578$ of the circumference of a circle to the radius 1.



For this spiral, we have

$$u = \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int a^2\theta^2 d\theta = \frac{1}{6}a^2\theta^3 + C;$$

and, if the area begins when $\theta = 0$, $C = 0$, and $u = \frac{1}{6}a^2\theta^3$. When $\theta = 2\pi$, $u =$ area $PAB = \frac{4}{3}a^2\pi^3$ is the area described by the radius vector during the first revolution. In

the second revolution, the radius vector again describes this area, and also the area $PBA'B'$ included between the first and second spires. Hence the area $PBA'B'$ is measured by

$$\frac{1}{6}\,a^2\left\{(4\pi)^3-(2\pi)^3\right\}=\frac{28}{3}a^2\pi^3.$$

It is evident that during any, as the m^{th}, revolution, the radius vector describes the whole area out to the m^{th} spire, and that, to find this area, the integral

$$u=\frac{1}{2}\int a^2\theta^2 d\theta=\frac{1}{6}\,a^2\theta^3$$

must be taken between the limits $(m-1)2\pi$ and $2m\pi$, which will give for this area denoted by u''

$$u''=\frac{1}{6}\,a^2(m2\pi)^3-\frac{1}{6}\,a^2(m-1)^3(2\pi)^3$$

$$=\frac{1}{6}a^2(2\pi)^3\left\{m^3-(m-1)^3\right\}.$$

In like manner, we have for the entire area denoted by u', out to the $(m-1)^{\text{th}}$ spire,

$$u'=\frac{1}{6}a^2(2\pi)^3\left\{(m-1)^3-(m-2)^3\right\}:$$

$$\therefore\qquad u''-u'=\frac{1}{6}a^2(2\pi)^3\left\{m^3-2\,(m-1)^3+(m-2)^3\right\},$$

which is the expression for the area included between the $(m-1)^{\text{th}}$ and the m^{th} spires.

If we suppose $a=\dfrac{1}{2\pi}$, this formula becomes

$$u''-u'=\frac{1}{6}\,2\pi\left\{m^3-2\,(m-1)^3+(m-2)^3\right\}$$

$$=\frac{m^3-2\,(m-1)^3+(m-2)^3}{3}\,\pi=(m-1)\,2\pi;$$

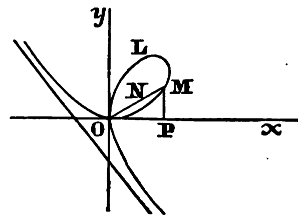
and in this, making $m=2$, we find 2π for the area included between the 1st and 2d spires. Hence the area included be-

tween the $(m-1)^{\text{th}}$ and m^{th} spires is $m-1$ times that included between the 1^{st} and 2^{d} spires.

239. The quadrature of curvilinear areas is sometimes facilitated by transforming rectilinear into polar co-ordinates.

Take, for example, the *folium of Descartes*, which, referred to rectangular axes, is represented by the equation

$$x^3 + y^3 - axy = 0.$$



This curve is composed of two branches, infinite in extent, which intersect at the origin of co-ordinates, and which have for a common asymptote the straight line of which the equation is

$$x + y + \frac{a}{3} = 0.$$

To determine the area of any portion of this curve in terms of the primitive co-ordinates, we must find what the integral of ydx becomes when in it the value of y derived from the equation of the curve is substituted. This requires the solution of an equation of the third degree; but if rectilinear be changed into polar co-ordinates, the pole being at O, there will be but one value of the radius vector in any assumed direction; for, the origin being a double point, two values of r, each equal to zero, must satisfy the polar equation of the curve, and the first member of this equation must be divisible by r^2.

Ox being the polar axis, the transformed equation is

$$r^3(\cos.^3\theta + \sin.^3\theta) - a\gamma^2 \sin.\theta \cos.\theta = 0:$$

whence
$$r = \frac{a \sin.\theta \cos.\theta}{\sin.^3\theta + \cos.^3\theta}.$$

For the area of the segment OMN, we have $u = \frac{1}{2}\int_0^\theta r^2 d\theta$,

which, by substituting for r its value above, becomes

$$u = \frac{1}{2}\int_0^\theta \frac{a^2 \sin.^2\theta \cos.^2\theta}{(\cos.^3\theta + \sin.^3\theta)^2} d\theta = \frac{a^2}{2}\int_0^\theta \frac{\sin.^2\theta d\theta}{\cos.^4\theta(1 + \tan.^3\theta)^2}$$

$$= \frac{a^2}{2}\int_0^\theta \frac{\tan.^2\theta \frac{d\theta}{\cos.^2\theta}}{(1 + \tan.^3\theta)^2}.$$

To effect this integration, put

$$1 + \tan.^3\theta = z: \therefore dz = 3\tan.^2\theta \frac{d\theta}{\cos.^2\theta},$$

and hence

$$\int \frac{\tan.^2\theta \frac{d\theta}{\cos.^2\theta}}{(1 + \tan.^3\theta)^2} = \frac{1}{3}\int \frac{dz}{z^2} = -\frac{1}{3z} + C$$

$$= -\frac{1}{3(1 + \tan.^3\theta)} + C:$$

$$\therefore \qquad u = -\frac{a^2}{6}\frac{1}{1 + \tan.^3\theta} + C.$$

The area beginning when $\theta = 0$, we have $C = \frac{a^2}{6}$, and consequently

$$u = \frac{a^2}{6}\frac{\tan.^3\theta}{1 + \tan.^3\theta}.$$

The entire area OML is found by making $\theta = \frac{\pi}{2}$ in the value of u, which then becomes $\frac{a^2}{6}$; for then the fraction $\frac{\tan.^3\theta}{1 + \tan.^3\theta} = 1$.

SECTION VI.

RECTIFICATION OF PLANE CURVES.

240. THE rectification of a curve is the operation of finding its length, and the curve is said to be *rectifiable* when this length can be represented by a straight line.

Denoting by s the arc of a curve comprised between a fixed point and an arbitrary point (x, y), we have

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \frac{dy^2}{dx^2}} \text{ (Art. 161);}$$

and, by integration,

$$s = \int dx\sqrt{1 + \frac{dy^2}{dx^2}}.$$

By means of the equation of the curve, ds may be expressed in terms of either x or y; and, the integral being then taken between the assigned limits, we have the length of the curve.

EXAMPLE 1. ***The Common Parabola.*** From the equation $y^2 = 2px$ of the curve, we find $ydy = pdx$, $dx = \frac{ydy}{p}$. This value of dx, substituted in the differential formula, gives

$$ds = \sqrt{\frac{y^2dy^2}{p^2} + dy^2} = \frac{dy}{p}\sqrt{y^2 + p^2}:$$

whence, making the arc begin at the vertex of the parabola,

$$s = \frac{1}{p}\int_0^y dy\sqrt{y^2 + p^2}$$

$$= \frac{y}{2p}\sqrt{y^2 + p^2} + \frac{p}{2}l(y + \sqrt{y^2 + p^2}) + C,$$

by Ex. 5, page 327.

Since the integral is to be zero, for $y=0$ we have

$$0=\frac{p}{2}lp+C:\ \therefore\ C=-\frac{p}{2}lp.$$

By the substitution of this value of C, the formula becomes

$$s=\frac{y}{2p}\sqrt{y^2+p^2}+\frac{p}{2}l\left(\frac{y+\sqrt{y^2+p^2}}{p}\right).$$

Ex. 2. ***The Ellipse.*** From the equation of the curve

$$a^2y^2+b^2x^2=a^2b^2,$$

we get

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}:$$

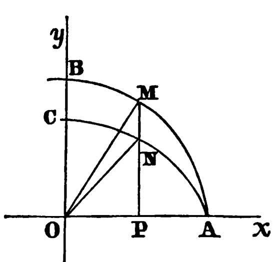
hence

$$ds=dx\sqrt{1+\frac{b^4x^2}{a^4y^2}}=dx\sqrt{1+\frac{b^4x^2}{a^2(a^2b^2-b^2x^2)}},$$

or $$ds=dx\sqrt{\frac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}}=dx\sqrt{\frac{a^2-e^2x^2}{a^2-x^2}},$$

in which $e=\frac{\sqrt{a^2-b^2}}{a}$ is the eccentricity of the ellipse.

Suppose the arc CN to be estimated from the vertex C of the minor axis; then, to get the length of the arc CNA, the integral of the expression for ds must be taken between the limits $x=0$ and $x=a$; but all the values of x between 0 and a will be given by $x=a\sin.\varphi$, the angle φ varying between 0 and $\frac{\pi}{2}$. The substitution in the value of ds of these values of x and its differential gives



$$ds=a\sqrt{1-e^2\sin.^2\varphi}\,d\varphi;$$

and therefore

$$s = CN = a\int_0^{\varphi} \sqrt{1 - e^2 \sin.^2 \varphi}\, d\varphi.$$

This integral belongs to a class of functions for which we have no expression except under the sign of integration; and, to find its approximate value, we must have recourse to a series. The Binomial Formula gives

$$(1 - e^2 \sin.^2 \varphi)^{\frac{1}{2}} = 1 - \frac{1}{2} e^2 \sin.^2 \varphi - \frac{1}{2}\frac{1}{4} e^4 \sin.^4 \varphi$$
$$- \frac{1}{2}\frac{1}{4}\frac{3}{6} e^6 \sin.^6 \varphi - \frac{1}{2}\frac{1}{4}\frac{3}{6}\frac{5}{8} e^8 \sin.^8 \varphi \ldots :$$

hence, for the arc CN, we have

$$s = a\varphi - \frac{1}{2} ae^2 \int \sin.^2 \varphi d\varphi - \frac{1}{2}\frac{1}{4} ae^4 \int \sin.^4 \varphi d\varphi$$
$$- \frac{1}{2}\frac{1}{4}\frac{3}{6} ae^6 \int \sin.^6 \varphi d\varphi \ldots.$$

The integrals in the second member of this value of s may be found by applying Formula 1 of Art. 221. We should thus get, by taking all the integrals between the limits 0 and $\frac{\pi}{2}$,

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin.^2 \varphi}\, d\varphi = \frac{\pi}{2} - \frac{1}{2} e^2 \frac{\pi}{2} - \frac{1}{2}\frac{1}{4} e^4 \frac{1}{2}\frac{3}{4}\frac{\pi}{2}$$
$$- \frac{1}{2}\frac{1}{4}\frac{3}{6} e^6 \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{\pi}{2} \ldots :$$

hence, for the arc CNA, we have

$$CNA = \frac{\pi a}{2} \left\{ 1 - \left(\frac{1}{2} e\right)^2 - \frac{1}{3}\left(\frac{1}{2}\frac{3}{4} e^2\right)^2 - \frac{1}{5}\left(\frac{1}{2}\frac{3}{4}\frac{5}{6} e^3\right)^2 \right.$$
$$\left. - \frac{1}{7}\left(\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8} e^4\right)^2 \right\}.$$

This is a converging series, and the more rapidly so as e becomes less, or as a and b approach equality. When the

eccentricity is very small, it would be sufficient to compute but a few of the terms of the series.

This value of the arc CNA may be found without using Formula 1 of Art. 221; for, assuming the first equation in that article, and taking the integral between the limits 0 and $\frac{\pi}{2}$, we have

$$\int_0^{\frac{\pi}{2}} \sin.^m \varphi d\varphi = \frac{m-1}{m} \int_0^{\frac{\pi}{2}} \sin.^{m-2} \varphi d\varphi.$$

In like manner,

$$\int_0^{\frac{\pi}{2}} \sin.^{m-2} \varphi d\varphi = \frac{m-3}{m-2} \int_0^{\frac{\pi}{2}} \sin.^{m-4} \varphi d\varphi,$$

$$\int_0^{\frac{\pi}{2}} \sin.^{m-4} \varphi d\varphi = \frac{m-5}{m-4} \int_0^{\frac{\pi}{2}} \sin.^{m-6} \varphi d\varphi,$$

. . . .

$$\int_0^{\frac{\pi}{2}} \sin.^2 \varphi d\varphi = \frac{1}{2} \frac{\pi}{2}.$$

Multiplying these equations member by member, there results

$$\int_0^{\frac{\pi}{2}} \sin.^m \varphi d\varphi = \frac{(m-1)(m-3)(m-5)\ldots 3.1}{m(m-2)(m-4)\ldots 4.2} \frac{\pi}{2}.$$

The values given by this, by making m equal to 2, 4, 6,..., successively substituted in the value of s, lead to the result before found.

The angle φ is found by the following construction: On the major axis as a diameter describe a circle; produce the ordinate PN to meet the circumference at M, and draw OM; then

$$x = OP = OM \cos. POM = a \sin. BOM:$$

hence $\varphi =$ angle BOM.

Ex. 3. ***The Hyperbola.*** Assuming the equation

$$a^2y^2 - b^2x^2 = -b^2a^2$$

of this curve, and proceeding as in the case of the ellipse, we get

$$ds = dx\sqrt{\frac{(a^2+b^2)x^2-a^4}{a^2(x^2-a^2)}}.$$

To simplify, put $\sqrt{a^2+b^2}=ae$; then

$$ds = dx\sqrt{\frac{e^2x^2-a^2}{x^2-a^2}}.$$

Now, for one branch of the hyperbola, all admissible values of x are comprised between $+a$ and $+\infty$, and for the other branch such values are comprised between $-a$ and $-\infty$; and it is evident that all of these values will be given by the equation $x=\frac{a}{\cos.\varphi}$ by making φ vary between 0 and $\frac{\pi}{2}$ for one branch, and between $\frac{\pi}{2}$ and π for the other.

Substituting this value of x, we have

$$dx = \frac{a\sin.\varphi d\varphi}{\cos.^2\varphi},$$

and $$ds = \frac{\sqrt{a^2e^2-a^2\cos.^2\varphi}}{\cos.^2\varphi}d\varphi = \frac{ae}{\cos.^2\varphi}\sqrt{1-\frac{\cos.^2\varphi}{e^2}}\,d\varphi:$$

whence (fig. Ex. 5, p. 399)

$$s = AM = \int_0^\varphi ae\frac{1}{\cos.^2\varphi}\sqrt{1-\frac{\cos.^2\varphi}{e^2}}\,d\varphi.$$

Developing the radical in this integral, we get

$$s = ae\int_0^\varphi \frac{1}{\cos.^2\varphi}\left\{1-\frac{1}{2}\frac{\cos.^2\varphi}{e^2}-\frac{1}{2}\frac{1}{4}\frac{\cos.^4\varphi}{e^4}-\dots\right.$$
$$\left.-\frac{1.1.3.5\dots(2n-3)}{2.4.6\dots 2n}\frac{\cos.^{2n}\varphi}{e^{2n}}\right\}d\varphi,$$

or $$s = ae\tan.\varphi-\frac{1}{2}\frac{a}{e}\varphi-$$

$$\frac{a}{e}\int_0^\varphi\left\{\frac{1}{2}\frac{1}{4}\frac{\cos.^2\varphi}{e^2}+\frac{1}{2}\frac{1}{4}\frac{3}{6}\frac{\cos.^{\cdot}\varphi}{e^4}+\dots\right\}d\varphi.$$

The integration now depends on that of expressions of the form $\cos.^m \varphi d\varphi$, and may be effected by the application of Formula 1, Art. 221, after changing in it x into $\frac{\pi}{2} - \varphi$.

Ex. 4. ***The Cycloid.*** The differential equation of this curve (Art. 146) is

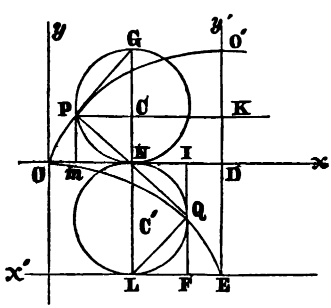
$$dx = dy\sqrt{\frac{y}{2r-y}}, \text{ or } dy = dx\sqrt{\frac{2r-y}{y}}.$$

In the formula $ds = \sqrt{dx^2 + dy^2}$, replacing dx by its value, we get

$$ds = \sqrt{2r}(2r-y)^{-\frac{1}{2}}dy:$$

$$\therefore \quad s = -2\sqrt{2r(2r-y)} + C.$$

If s be estimated from the origin O to the right, we must have



$$0 = -4r + C: \therefore C = 4r,$$

and

$$s = OP = 4r - 2\sqrt{2r(2r-y)}.$$

In this, making $y = 2r$, we have OO', the semi-arc of the cycloid, equal to $4r$, and the whole arc therefore equal to $8r$, or four times the diameter of the generating circle.

Estimating the arc from the vertex O' to the left, then $C = 0$, since at this point $y = 2r$; and we have

$$O'P = -2\sqrt{2r(2r-y)}.$$

But $$\sqrt{2r(2r-y)} = \sqrt{GN \times GC} = PG:$$

hence arc $O'P = 2$ chord PG; that is, the length of the arc of a cycloid, estimated from the vertex, is twice the corresponding chord of the generating circle.

SECTION VII.

DOUBLE INTEGRATION. — TRIPLE INTEGRATION.

241. ***Double Integrals*** are expressions involving two integrals with respect to different variables. Suppose it is required to find the value of u which will satisfy the equation $\frac{d^2u}{dydx} = \varphi(x, y)$, the variables x and y being independent.

This equation may be written

$$\frac{d}{dy}\left(\frac{du}{dx}\right) = \varphi(x, y) = \frac{dv}{dy},$$

by making $v = \frac{du}{dx}$. The function v must be such, that its differential co-efficient with respect to y, x being considered as constant, is equal $\varphi(x, y)$. We therefore have

$$v = \int \varphi(x, y) dy = \frac{du}{dx}:$$

hence u must be such a function of x and y that its differential co-efficient with respect to x, y being constant, is equal to $\int \varphi(x, y) dy$; and therefore

$$u = \int \left\{ \int \varphi(x, y) dy \right\} dx.$$

The value of u is thus obtained by integrating the original expression with respect to y, and then integrating the result with respect to x.

The last equation is generally and more concisely written

$$u = \int\int \varphi(x, y) dxdy, \text{ or } u = \int\int \varphi(x, y) dydx;$$

the first form indicating that the first integration is performed with respect to x, and the second integration with respect to y. The second form indicates that the order of integration is reversed.

242. It was shown (Art. 91) that $\frac{d^2u}{dxdy}=\frac{d^2u}{dydx}$, or that these partial differential co-efficients were the same, in whichever order, with respect to x and y, the differentiation is performed. We will now prove that the result of the integration in the one order can differ from that obtained in the other only by the sum of two arbitrary functions, the one of x, and the other of y. Let u_1, u_2, be two functions of x and y, either of which satisfies the equation $\frac{d^2u}{dxdy}=\varphi(x, y)$; then

$$\frac{d^2u_1}{dxdy}=\varphi(x, y), \quad \frac{d^2u_2}{dxdy}=\varphi(x, y):$$

$$\therefore \qquad \frac{d^2u_1}{dxdy}-\frac{d^2u_2}{dxdy}=0,$$

or
$$\frac{d}{dx}\left(\frac{dv}{dy}\right)=0, \text{ putting } v=u_1-u_2.$$

Now, $\frac{dv}{dy}$ cannot be a function of x, otherwise its differential co-efficient with respect to x could not be 0; but it may be any function of y. Hence we may put

$$\frac{dv}{dy}=f(y); \text{ whence } v=\int f(y)dy+\chi(x),$$

in which $\chi(x)$ denotes an arbitrary function of x. Putting $\int f(y)dy=\psi(y)$, $\psi(y)$ being as arbitrary as $f(y)$, we have finally

$$v=u_1-u_2=\psi(y)+\chi(x),$$

as it was proposed to prove.

243. A double integral $\int_a^b \int_\alpha^\beta \varphi(x, y)\,dxdy$ is the limit of all the products of the form $\varphi(x, y)\Delta x \Delta y$ between the limits of integration. Let $\varphi(x, y)$ be a function of x and y, which remains finite and continuous for values of x between a and b, and for values of y between α and β.

To abbreviate, put $\varphi(x, y) = z$. Now, if we suppose x to be constant while y varies between the limits α and β, we have (Art. 192)

$$\int_\alpha^\beta z\,dy = \lim. \Sigma z \Delta y.$$

Multiplying both members of this equation by Δx, and supposing x to vary between the limits a and b while y remains constant, there results

$$\Sigma \Delta x \int_\alpha^\beta z\,dy = \Sigma \Delta x \lim. \Sigma z \Delta y:$$

hence $\lim. \Sigma \Delta x \int_\alpha^\beta z\,dy = \lim. \Sigma \Delta x \lim. \Sigma z \Delta y = \lim. \Sigma\Sigma z \Delta x \Delta y.$

But $$\lim. \Sigma \Delta x \int_\alpha^\beta z\,dy = \int_a^b \int_\alpha^\beta z\,dxdy$$

by the article above referred to: therefore

$$\int_a^b \int_\alpha^\beta \varphi(x, y)\,dxdy = \lim. \Sigma\Sigma \varphi(x, y) \Delta x \Delta y.$$

Writers do not agree as to the notation for double integrals; some making the first sign $\int$ refer to the variable whose differential comes first in the integral, while others make the first sign $\int$ refer to the other variable. In what follows, the first sign $\int$ will relate to the variable whose differential is first written in the indicated integral.

244. In the last article, it was supposed that the variables x and y were independent. It is sometimes the case, however, that the limits in the first integration are functions of the other variable. For example, let $\int_a^b \int_\alpha^\beta \varphi(x, y)\,dxdy$ be the required integral in which $\alpha = \chi(x)$, and $\beta = \psi(x)$; then

$$\int_a^b \int_\alpha^\beta \varphi(x, y)\,dxdy = \int_a^b \int_{\chi(x)}^{\psi(x)} \varphi(x, y)\,dxdy.$$

Suppose $F(x, y)$ to be the result obtained by integrating, first with respect to y, regarding x as constant; then, for the integral between the assigned limits for y, we have

$$F\{x, \psi(x)\} - F\{x, \chi(x)\},$$

and finally

$$\int_a^b \int_{\chi(x)}^{\psi(x)} \varphi(x, y)\,dxdy = \int_a^b \Big(F\{x, \psi(x)\} - F\{x, \chi(x)\}\Big)dx.$$

When the limits of a double integral are constant, it is immaterial in what order, with respect to the variables, the integration is effected; that is, a change in the order of integration does not require a change in the values of the limits. But when the limits for one variable are functions of the other variable, and the order of integration is changed, a special investigation is necessary to determine what the new limits must be to preserve the equality of the results. A geometrical illustration of this will be given in the next section.

245. ***Triple Integration.*** Let it be required to determine a function u of the three independent variables x, y, z, which will satisfy the equation $\dfrac{d^3u}{dxdydz} = V$. We may write

$$\frac{d^3u}{dxdydz} = \frac{d}{dz}\,\frac{d^2u}{dxdy} = V,$$

or

$$\frac{d^3u}{dxdydz}\,dz = \frac{d}{dz}\,\frac{d^2u}{dxdy}\,dz = Vdz:$$

hence by integration with respect to z, regarding x and y as constant,

$$\frac{d^2u}{dxdy}=\int Vdz+T'';$$

T'' being an arbitrary function of x and y. Again: we have

$$\frac{d^2u}{dxdy}=\frac{d}{dy}\frac{du}{dx}=\int Vdz+T'',$$

or
$$\frac{d^2u}{dxdy}dy=\frac{d}{dy}\frac{du}{dx}dy=dy\int Vdz+T''dy,$$

which, by integrating with respect to y, x and z being constant, gives

$$\frac{du}{dx}=\int dy\int Vdz+T'+S';$$

T' being an arbitrary function of x and y coming from $\int T''dy$, and S' an arbitrary function of x and z.

Finally

$$u=\int\frac{du}{dx}dx=\int dx\int dy\int Vdz+T+S+R;$$

T, R, and S being arbitrary functions, — the first of x and y resulting from $\int T'dx$, the second of x and z resulting from $\int S'dx$, and the third of y and z.

It is usual to write the differentials together after the last sign of integration: the above equation thus becomes

$$u=\int\int\int Vdxdydz+T+S+R.$$

This example suffices to show the manner of passing from a differential co-efficient of any order of a function of several variables back to the function itself. When the variables are independent of each other, as has been here supposed, there

is no dependence between the arbitrary functions T, S, R; but more commonly at the limits of the integral the variables are not independent of each other. For example, the limits of the integral with respect to z may correspond to $z = F(x, y)$, $z = F_1(x, y)$; those with respect to y, to $y = f(x)$, $y = f_1(x)$; and, finally, those with respect to x, to $x = a$, $x = b$.

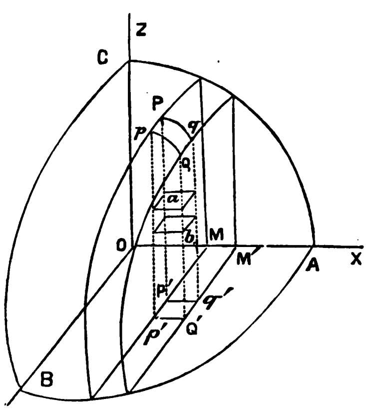
By a demonstration similar to that given in the case of a double integral (Art. 244), it may be shown that

$$\int_a^b dx \int_\alpha^\beta dy \int_\delta^\gamma \varphi(x, y, z)\, dz = \text{lim.}\, \Sigma\Sigma\Sigma \Delta x \Delta y \Delta z.$$

SECTION VIII.

QUADRATURE OF CURVED SURFACES. — CUBATURE OF SOLIDS.

246. Let $F(x, y, z) = 0$ be the equation of any surface whatever, and take on this surface the point P, (x, y, z), and the adjacent point Q, $(x + \Delta x, y + \Delta y, z + \Delta z)$. Project these points in P', Q', on the plane x, y, and construct the rectangle $P'Q'$ by drawing parallels to the axes Ox, Oy. The lateral faces of the right prism of which $P'Q'$ is the base will intercept the element PQ of the curved surface. Denote by λ the angle that the tangent plane to the surface at the point P makes with the plane (x, y). This plane is determined by the tangent lines drawn to the curves Pq, Pp, at the point P. The tangent line to the first curve makes with the axis of x an angle of which $\dfrac{dz}{dx}$ is the tangent, and the tangent line to the second makes with the axis of y an angle of which $\dfrac{dz}{dy}$ is the tangent. These are the angles which the traces of the plane of these two lines, that is, of the tangent plane to the surface at the point P on the planes (z, x), (z, y), make with the same axes. Now, from



propositions 1 and 3, chap. ix., Robinson's "Analytical Geometry," we readily find, without regard to sign,

$$\cos.\lambda = \frac{1}{\left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{1}{2}}}.$$

The rectangle $P'Q'$ is measured by $\Delta x \Delta y$, and is the projection on the plane (x, y) of the corresponding element of the tangent plane. This element is measured by $\frac{\Delta x \Delta y}{\cos.\lambda}$: hence, for the element of the tangent plane, we have

$$\frac{\Delta x \Delta y}{\cos.\lambda} = \left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{1}{2}} \Delta x \Delta y$$

$$= \sec.\lambda \Delta x \Delta y.$$

Let S denote any extent of the surface under consideration, and assume that the limit of the sum of the terms $\sec.\lambda \Delta x \Delta y$, for all values of x and y between assigned limits, is the area of the surface; then

$$S = \int\int \left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{1}{2}} dxdy.$$

If the surface is limited by two planes parallel to the plane (z, y) at the distances $x = a$, $x = b$, and by the surfaces of two right cylinders whose bases are represented by the equations $y = \varphi(x)$, $y = \psi(x)$, we should have

$$S = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} \left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{1}{2}} dy$$

and, when the cylindrical surfaces reduce to planes parallel to the plane (zx), $\varphi(x)$ and $\psi(x)$ become constants c and e, and the formula reduces to

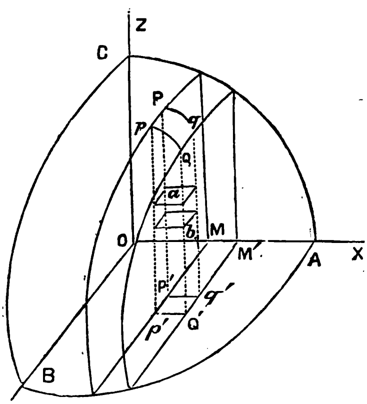
$$S = \int_a^b dx \int_c^e \left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{1}{2}} dy.$$

247. *Area of Surfaces of Revolution.* If $y=f(x)$ be the equation of a curve referred to rectangular axes, the differential co-efficient of the area of the surface generated by the revolution of this curve about the axis of x has been found (Art. 167) to be

$$\frac{dS}{dx}=2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}:$$

$$\therefore \qquad S=2\pi\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,ydx.$$

248. *Volumes of Solids.* Consider the volume bounded by the surface of which $F(x, y, z)=0$ is the equation; and through the point P, (x, y, z), in this surface, pass planes parallel to the planes (z, x), (z, y); and also through the point Q, $(x+\Delta x, y+\Delta y, z+\Delta z)$, adjacent to the point P, pass planes parallel to the same co-ordinate planes. These four planes are the lateral boundaries of a prismatic column, having $P'Q'$ for its base, and terminated above by the element PQ of the curved surface. The volume of this column is measured by $z\Delta x\Delta y$,



when Δx, Δy, are decreased without limit; and the volume bounded by any portion of the curved surface, the plane (x, y) and planes parallel to the planes (z, y), (z, x), will be the limit of the sum of a series of terms of which $z\Delta x\Delta y$ is the type. Denoting this volume by V, we have

$$V=\Sigma z\Delta x\Delta y=\int\int zdxdy.$$

From the equation $F(x, y, z) = 0$, which is the equation of the surface, we have $z = \varphi(x, y)$. If we integrate first with respect to y, we get the sum of the columns forming a layer, included between two planes perpendicular to the axis of x; and hence the limits of integration with respect to y become functions of x, and we should have $\int z dy = f(x)$; $f(x)$ being, in fact, the area of the section of the solid made by a plane parallel to the plane (z, y). Thence, finally, $V = \int f(x) dx$.

249. *Volumes of Solids of Revolution.* The differential co-efficient of the volume generated by the revolution, about the axis of x, of the plane area bounded on the one side by the axis of x, and on the other by the curve having $y = f(x)$ for its equation, has been found (Art. 166) to be

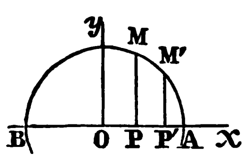
$$\frac{dV}{dx} = \pi y^2 = f(x):$$

hence, by integration,

$$V = \pi \int y^2 dx = \int f(x) dx.$$

Here, as was the case at the end of the last article, $f(x) = \pi y^2$ is the area of a section of the solid made by a plane perpendicular to the axis of x; and the integral is the expression for the sum of the elementary slices into which we may conceive the solid to be divided by such planes.

APPLICATIONS.



EXAMPLE 1. Required the measure of the zone generated by the revolution of the arc MM' of a circle about the diameter BA. The equation of the circle is $x^2 + y^2 = R^2$. Denoting the area of the zone by S, if $OP = a$, $OP' = b$, we shall have (Art. 247)

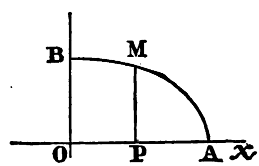
$$S = 2\pi \int_a^o y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= 2\pi \int_a^b y \sqrt{1 + \frac{x^2}{y^2}} dx = 2\pi \int_a^b R dx$$

$$= 2\pi R (b - a) = 2\pi R \times PP'.$$

To get the entire surface of the sphere, the integral must be taken between the limits $x = -R$, $x = R$, which will give $S = 4\pi R^2$.

Ex. 2. Suppose the ellipse of which the arc BMA is a quadrant to revolve about its transverse axis: required the measure of the surface generated by the portion BM of this arc, beginning at the extremity of the conjugate axis. We now have



$$S = 2\pi \int_0^x y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

From the equation of the ellipse, $a^2y^2 + b^2x^2 = a^2b^2$, we get $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$; whence

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{a^4y^2 + b^4x^2}}{a^2y} = \frac{b\sqrt{a^4 - (a^2 - b^2)x}}{a^2y};$$

and finally, by making $\sqrt{a^2 - b^2} = ae$, we have

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{b\sqrt{a^2 - e^2x^2}}{ay}:$$

therefore

$$S = 2\pi \frac{b}{a} \int_0^x \sqrt{a^2 - e^2x^2}\, dx = \frac{2\pi be}{a} \int_0^x \sqrt{\frac{a^2}{e^2} - x^2}\, dx.$$

But (Ex. 2, p. 326)

$$\int_0^x \sqrt{\frac{a^2}{e^2} - x^2}\,dx = \frac{1}{2} x \sqrt{\frac{a^2}{e^2} - x^2} + \frac{1}{2}\frac{a^2}{e^2}\sin.^{-1}\frac{ex}{a}:$$

therefore

$$S = \frac{\pi b e}{a}\left(x\sqrt{\frac{a^2}{e^2} - x^2} + \frac{a^2}{e^2}\sin.^{-1}\frac{ex}{a}\right).$$

If, in this expression, we make $x = a$, and take twice the result, we get

$$S = 2\pi b^2 + \frac{2\pi b a}{e}\sin.^{-1} e$$

for the entire surface of the prolate ellipsoid of revolution.

Suppose, now, that $a < b$, or that the ellipse is revolved about its conjugate axis, and put $\sqrt{b^2 - a^2} = be$; then we shall have

$$S = 2\pi\int_0^x y\,\frac{b\sqrt{a^4 + (b^2 - a^2)x^2}}{a^2 y}\,dx$$

$$= \frac{2\pi b}{a^2}\int_0^x \sqrt{a^4 + b^2e^2x^2}\,dx = \frac{2\pi b^2 e}{a^2}\int_0^x \sqrt{\frac{a^4}{b^2e^2} + x^2}\,dx.$$

But (Ex. 5, p. 327)

$$\int_0^x \sqrt{\frac{a^4}{b^2e^2} + x^2}\,dx = \frac{1}{2}x\sqrt{\frac{a^4}{b^2e^2} + x^2} + \frac{a^4}{2b^2e^2}\,l\left(x + \sqrt{\frac{a^4}{b^2e^2} + x^2}\right):$$

therefore

$$S = \frac{\pi b^2 e}{a^2}\left\{x\sqrt{\frac{a^4}{b^2e^2} + x^2} + \frac{a^4}{b^2e^2}\,l\left(x + \sqrt{\frac{a^4}{b^2e^2} + x^2}\right)\right\} + C.$$

Since this integral should be zero, for $x = 0$ we have

$$C = -\frac{\pi b^2 e}{a^2}\,\frac{a^4}{b^2e^2}\,l\frac{a^2}{be} = -\frac{\pi a^2}{e}\,l\frac{a^2}{be}:$$

hence

$$S = \frac{\pi b^2 e}{a^2}\left\{x\sqrt{\frac{a^4}{b^2e^2} + x^2} + \frac{a^4}{b^2e^2}\,l\left(\frac{x + \sqrt{\frac{a^4}{b^2e^2} + x^2}}{\frac{a^2}{be}}\right)\right\}.$$

If in this we make $x = a$, and take twice the result, we shall have

$$S = 2\pi b\sqrt{a^2 + b^2e^2} + \frac{2\pi a^2}{e}\, l\left(\frac{be + \sqrt{a^2 + b^2e^2}}{a}\right)$$

for the entire surface of the oblate ellipsoid of revolution.

If we suppose $a = b$, and therefore $e = 0$, the second term in the last expression for S takes the form $\frac{0}{0}$; but, by the rule for the evaluation of indeterminate forms, we readily find

$$\lim. \frac{l\left(\frac{be + \sqrt{a^2 + b^2e^2}}{a}\right)}{e} = 1:$$

whence we have $4\pi a^2$ for the surface of the sphere.

Ex. 3. ***Cubature of the Ellipsoid of Revolution.*** The equation of the ellipse, referred to its major axis and the tangent line at its vertex, is $y^2 = \frac{b^2}{a^2}(2ax - x^2)$; and therefore, for the volume of the ellipsoid, we have (Art. 249)

$$V = \frac{\pi b^2}{a^2}\int_0^x (2ax - x^2)\,dx = \frac{\pi b^2}{a^2}\left(ax^2 - \frac{x^3}{3}\right).$$

To get the entire volume, we make $x = 2a$; and then

$$V = \frac{\pi b^2}{a^2}\left(4a^3 - \frac{8}{3}a^3\right) = \frac{4}{3}\pi b^2 a.$$

This is the volume of the prolate ellipsoid. To get that of the oblate ellipsoid, a and b must be interchanged in the last formula. We thus get, for the measure of the entire volume, $\frac{4}{3}\pi a^2 b$; from which it is seen that this volume is greater than the first. Making $a = b$, the ellipsoid becomes a sphere, the

volume of which is expressed by $\frac{4}{3}\pi a^3$; and, for the volume of a spherical segment of a single base, the expression is

$$\frac{\pi}{3}x^2(3a-x).$$

Ex. 4. ***Volume generated by the Revolution of a Cycloid about its Base.***

In the formula $V=\int \pi y^2 dx$, substitute for dx its value $dx=\frac{ydy}{(2ry-y^2)^{\frac{1}{2}}}$ derived from the equation of the cycloid. and we have

$$V=\int \frac{\pi y^3 dy}{(2ry-y^2)^{\frac{1}{2}}}+C;$$

but (Ex. 1, page 370)

$$\int \frac{y^3dy}{(2ry-y^2)^{\frac{1}{2}}}=-\frac{y^2}{3}(2ry-y^2)^{\frac{1}{2}}+\frac{5}{3}r\int \frac{y^2dy}{(2ry-y^2)^{\frac{1}{2}}},$$

$$\int \frac{y^2dy}{(2ry-y^2)^{\frac{1}{2}}}=-\frac{y}{2}(2ry-y^2)^{\frac{1}{2}}+\frac{3}{2}r\int \frac{ydy}{(2ry-y^2)^{\frac{1}{2}}},$$

$$\int \frac{ydy}{(2ry-y^2)^{\frac{1}{2}}}=-(2ry-y^2)^{\frac{1}{2}}+r\int \frac{dy}{(2ry-y^2)^{\frac{1}{2}}}$$

$$=-(2ry-y^2)^{\frac{1}{2}}+r\,\text{ver.sin.}^{-1}\frac{y}{r}:$$

therefore, by substitution and reduction,

$$V=\int \frac{\pi y^3dy}{(2ry-y^2)^{\frac{1}{2}}}=-\pi(2ry-y^2)^{\frac{1}{2}}\left(\frac{y^2}{3}+\frac{5}{6}ry+\frac{5}{2}r^2\right)$$

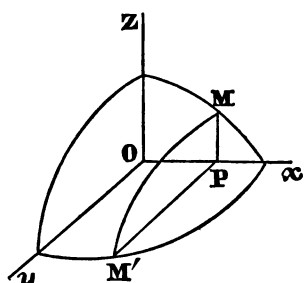
$$+\frac{5}{2}\pi r^3\,\text{ver.sin.}^{-1}\frac{y}{r}+C.$$

Taking this integral between the limits $y = 0$, $y = 2r$, and doubling the result, we have, for the entire volume generated by the revolution of a single branch of the cycloid,

$$V = 5\pi^2 r^3.$$

Ex. 5. ***Volume of an Ellipsoid.*** Take for the co-ordinate axes the *principal axes* of the ellipsoid. The equation of its surface is then $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

The section PMM' of the ellipsoid made by a plane parallel to the plane ZOy, and at the distance $OP = x$ from the origin, has for its equation



$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}.$$

The semi-axes of this section will be found by making in succession $z = 0$, $y = 0$; they are

$$PM' = b\sqrt{1 - \frac{x^2}{a^2}}, \quad PM = c\sqrt{1 - \frac{x^2}{a^2}}:$$

hence the area of the section is

$$\pi bc\left(1 - \frac{x^2}{a^2}\right) = \frac{\pi bc}{a^2}(a^2 - x^2);$$

and, for the volume of the segment included between the planes ZOy and PMM', we have

$$V = \frac{\pi bc}{a^2}\int_0^x (a^2 - x^2)\,dx = \frac{\pi bc}{a^2}\left(a^2 x - \frac{x^3}{3}\right).$$

To get the volume of half the ellipsoid, make in this formula $x = a$, which gives $V = \frac{2}{3}\pi abc$; and hence the entire volume is measured by $\frac{4}{3}\pi abc$.

Ex. 6. The areas of surfaces and volumes of solids have thus far been found by single integration. As an example of double integration, let it be required to find the volume bounded by the surface determined by the equation $xy = az$, and by the four planes having for their equations

$$x = x_1,\ x = x_2,\ y = y_1,\ y = y_2.$$

The expression for this volume is

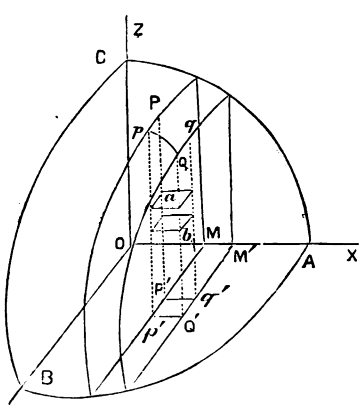
$$V = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\frac{xy}{a}\,dxdy = \frac{1}{4a}\left(y_2^2 - y_1^2\right)\left(x_2^2 - x_1^2\right)$$

$$= \frac{1}{4a}(x_2 - x_1)(y_2 - y_1)(x_1y_1 + x_2y_2 + x_1y_2 + x_2y_1)$$

$$= \frac{1}{4}(x_2 - x_1)(y_2 - y_1)(z_1 + z_2 + z_3 + z_4),$$

in which z_1, z_2, z_3, z_4, are the ordinates of the points in which the lateral edges of the volume considered pierce the surface $xy = az$.

Ex. 7. To illustrate triple integration geometrically, in the figure suppose planes to be passed perpendicular to the axis of z. Let two of these planes be at the distances z and $z + \Delta z$ respectively from the origin of co-ordinates; cutting from the elementary column PQ' a rectangular parellelopipedon ab measured by $\Delta x \Delta y \Delta z$. This parallelopipedon may be considered as an element of the whole volume V: hence



$$V = \int\int\int dxdydz.$$

Required the portion of the volume of the right cylinder that is intercepted by the planes $z = x\tan.\theta$, $z = x\tan.\theta'$; the equation of the base of the cylinder being $x^2 + y^2 - 2ax = 0$. Here the limits of the integral are $z = x\tan.\theta$, $z = x\tan.\theta'$, $y = -\sqrt{2ax - x^2}$, $y = +\sqrt{2ax - x^2}$, $x = 0$, $x = 2a$: therefore, denoting the values of y by $-y_1$, $+y_1$,

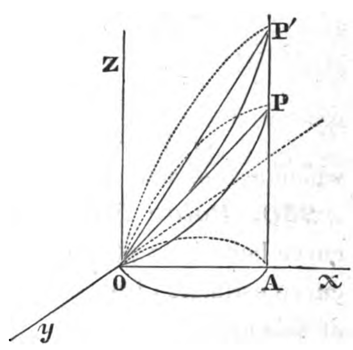
$$V = \int_0^{2a}\int_{-y_1}^{+y_1}\int_{x\tan.\theta}^{x\tan.\theta'} dxdydz$$

$$= \int_0^{2a}\int_{-y_1}^{+y_1} (\tan.\theta' - \tan.\theta)\,xdxdy$$

$$= 2(\tan.\theta' - \tan.\theta)\int_0^{2a} x\sqrt{2ax - x^2}\,dx$$

$$= (\tan.\theta' - \tan.\theta)\pi a^3.$$

The base of this cylinder is a circle in the plane (x, y) tangent to the axis of y at the origin of co-ordinates; and the secant planes pass through the origin, and are perpendicular to the plane (z, x). The required volume is therefore the portion of the cylinder included between the sections OP, OP'.



It can be seen from this example why, as was observed in Art. 244, when there is a relation between the variables at the limits of an integral, the order of integration cannot be changed without at the same time ascertaining if it be not necessary to make a corresponding change in the limiting values of the variables. In this case, after integrating with respect to z, we integrate with respect to y, taking the integral between the limits $y = -(2ax - x^2)^{\frac{1}{2}}$, $y = +(2ax - x^2)^{\frac{1}{2}}$;

that is, the integral is considered as bounded by the circumference of a circle tangent to the axis of y at the origin; but by what portion of the circumference is not specified until the limiting values of x are assigned. The integral with respect to x is then taken from $x = 0$ to $x = 2a$, which thus embraces the whole circumference.

But it is obvious, that, if the order of integration with respect to x and y be reversed, then, that the integral may embrace the whole base of the cylinder, the limits with respect to x must be $x = a - \sqrt{a^2 - y^2}$, $x = a + \sqrt{a^2 - y^2}$; and those with respect to y must be $y = -a$, $y = +a$. We now have, denoting the limiting values of x by x_1, $-x_1$,

$$V = \int_{-a}^{a} \int_{-x_1}^{x_1} \int_{x \tan.\theta}^{x \tan.\theta'} dydxdz$$

$$= \int_{-a}^{a} \int_{-x_1}^{x_1} (\tan.\theta' - \tan.\theta) xdydx$$

$$= 2a \int_{-a}^{a} (\tan.\theta' - \tan.\theta) \sqrt{a^2 - y^2} dy$$

$$= (\tan.\theta' - \tan.\theta) \pi a^3 \text{ (Ex. 2, page 326)};$$

which agrees with the first result.

250. Polar Formula. The polar equation of a plane curve being $r = \varphi(\theta)$, if s denote the length of an arc of the curve estimated from a fixed point, the differential co-efficient of this arc (Art. 163) is

$$\frac{ds}{d\theta} = \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}}:$$

$$\therefore \quad s = \int \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}} d\theta \quad (1);$$

or, by taking r as the independent variable,

$$s = \int \left\{ r^2 \left(\frac{d\theta}{dr}\right)^2 + 1 \right\}^{\frac{1}{2}} dr \quad (2).$$

EXAMPLE 1. Applying Formula 1 to the spiral of Archimedes, the equation of which is $r = a\theta$, we have

$$s = \int (r^2 + a^2)^{\frac{1}{2}} d\theta = a \int (1 + \theta^2)^{\frac{1}{2}} d\theta$$

$$= \frac{a\theta}{2}(1 + \theta^2)^{\frac{1}{2}} + \frac{a}{2}\, l \left\{\theta + (1 + \theta^2)^{\frac{1}{2}}\right\} + C.$$

If the arc considered begins at the pole where $\theta = 0$, then $C = 0$.

Ex. 2. For the logarithmic spiral, we have $r = ba^{\theta}$, or $r = be^{\frac{\theta}{c}}$ by making $a = e^{\frac{1}{c}}$; $\therefore\ \theta = cl\frac{r}{b}$, and $\frac{d\theta}{dr} = \frac{c}{r}$: whence, by Formula 2,

$$s = \int \sqrt{(1 + c^2)}\, dr = \sqrt{(1 + c^2)}\, r + C.$$

If the limits of the integral correspond to the radii vectores r_0, r_1, the length of the arc is

$$s = \sqrt{(1 + c^2)}(r_1 - r_0).$$

Since $r\frac{d\theta}{dr}$ is the expression for the tangent of the angle made by the radius vector of the curve at any point and the tangent line at that point, we have, calling this angle α, $\tan.\alpha = c$; hence $\sec.\alpha = \sqrt{(1 + c^2)}$, and $\frac{ds}{dr} = \sec.\alpha$: therefore $s = r\sec.\alpha + C$, and the definite portion of the arc answering to r_0, r_1, is $(r_1 - r_0)\sec.\alpha$.

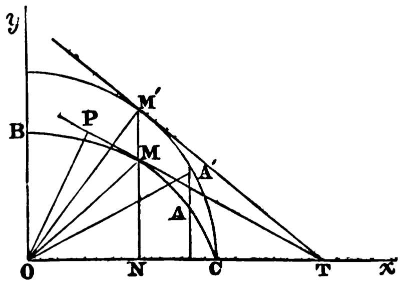
251. To find the length of a curve in terms of the radius vector and the perpendicular demitted from the pole to the tangent line to the curve at any point, we have $\cos.\alpha = \frac{dr}{ds}$ (cor. Art. 163): hence, if p denotes the length of the perpendicular,

$$\sin.\alpha = \frac{p}{r},\quad \cos.\alpha = \frac{\sqrt{r^2 - p^2}}{r};\quad \therefore\ \frac{dr}{ds} = \frac{\sqrt{r^2 - p^2}}{r};$$

therefore $\frac{ds}{dr} = \frac{r}{\sqrt{r^2 - p^2}}, \quad s = \int \frac{rdr}{\sqrt{r^2 - p^2}}.$

252. The length of a curve may also be expressed in terms of the perpendicular and its inclination to the initial line.

Let x and y be the co-ordinates of any point M of the curve, and denote by s the length of the curve included between the fixed point A and the point M. From the origin let fall the perpendicular OP upon the tangent to the curve at the point M, and make $OP = p$, $MP = u$, and the angle $POx = \theta$; then, from the figure, we readily find



$$p = x \cos.\theta + y \sin.\theta,$$
$$u = x \sin.\theta - y \cos.\theta.$$

Also we have

$$\frac{dy}{dx} = -\cot.\theta, \quad \frac{ds}{dx} = -\operatorname{cosec}.\theta:$$

therefore

$$\frac{dp}{d\theta} = -x \sin.\theta + y \cos.\theta + \cos.\theta \frac{dx}{d\theta} + \sin.\theta \frac{dy}{d\theta}.$$

But, since θ is the independent variable, $\frac{dy}{dx} = -\cot.\theta$ may be written $\dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{\cos.\theta}{\sin.\theta}$; whence

$$\sin.\theta \frac{dy}{d\theta} + \cos.\theta \frac{dx}{d\theta} = 0:$$

$$\therefore \qquad \frac{dp}{d\theta} = -x \sin.\theta + y \cos.\theta = -u,$$

$$\frac{d^2p}{d\theta^2} = -\frac{du}{d\theta} = -x \cos.\theta - y \sin.\theta - \sin.\theta \frac{dx}{d\theta} + \cos.\theta \frac{dy}{d\theta}.$$

But, from $\frac{dy}{dx} = -\cot.\theta$, we get

$$\cos.\theta \frac{dy}{d\theta} = -\cos.^2\theta \operatorname{cosec}.\theta \frac{dx}{d\theta}:$$

$$\therefore -\sin.\theta \frac{dx}{d\theta} + \cos.\theta \frac{dy}{d\theta} = -\sin.\theta \frac{dx}{d\theta} - \cos.^2\theta \operatorname{cosec}.\theta \frac{dx}{d\theta}$$

$$= -\frac{dx}{d\theta}\left(\frac{\sin.^2\theta + \cos.^2\theta}{\sin.\theta}\right) = -\operatorname{cosec}.\theta \frac{dx}{d\theta}.$$

The equation $\frac{ds}{dx} = -\operatorname{cosec}.\theta$ gives $\frac{ds}{d\theta} = \operatorname{cosec}.\theta \frac{dx}{d\theta}$; hence, by substitution,

$$\frac{d^2p}{d\theta^2} = -x \cos.\theta - y \sin.\theta + \frac{ds}{d\theta}$$

$$= -p + \frac{ds}{d\theta}:$$

therefore

$$\frac{dp}{d\theta} = -\int p d\theta + s;$$

$$\therefore \qquad s = \frac{dp}{d\theta} + \int p d\theta,$$

or

$$s + u = \int p d\theta.$$

Taking the integral between the limits $\theta_0, \theta_1, s_0, s_1, u_0, u_1$, being the corresponding values of s and u, we have

$$s_1 - s_0 + u_1 - u_0 = \int_{\theta_0}^{\theta_1} p d\theta.$$

The sign of u will be positive or negative according as the angle $POx = \theta$ is greater or less than the angle MOx. These

results may be used for several purposes, the most important of which are,—

First, To find the length of any portion of a curve, the equation of the curve being given. In this case, from the equation of the curve and the equation $\frac{dy}{dx} = -\cot.\theta$, x and y, and therefore $p = x\cos.\theta + y\sin.\theta$, can be determined in terms of θ; and, by integration, s may be found from the equation $s = \frac{dp}{d\theta} + \int p d\theta$.

Second, To find a curve, the length of a portion of which shall represent a proposed integral. Here, if the integral be $\int p d\theta$, p being a function of θ, the equation of the curve is found by eliminating θ between the equations

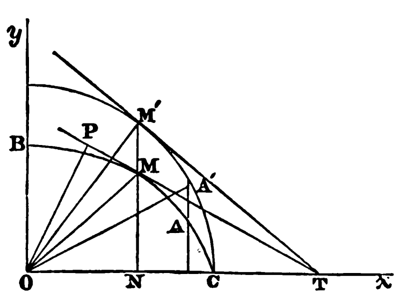
$$x = p\cos.\theta - \frac{dp}{d\theta}\sin.\theta,\quad y = p\sin.\theta + \frac{dp}{d\theta}\cos.\theta,$$

hich we get from the equations

$$p = x\cos.\theta + y\sin.\theta,\quad \frac{dp}{d\theta} = -x\sin.\theta + y\cos.\theta.$$

The proposed integral will then be represented by $s - \frac{dp}{d\theta}$.

APPLICATION.



Let BMC be a quadrant of an ellipse of which the equation referred to its centre and axes is

$$a^2y^2 + b^2x^2 = a^2b^2.$$

This equation, by making $b^2 = a^2(1 - e^2)$, may be put under the form

$$y^2 = (1 - e^2)(a^2 - x^2).$$

Make $POx = \theta$.

Then, from the properties of this curve, we have

$$\overline{OT}^2 = \frac{a^4}{x^2}, \quad \tan.^2 PTO = \cot.^2\theta = \frac{(1-e^2)x^2}{a^2 - x^2}.$$

From the last equation, we get

$$\sin.^2\theta = \frac{a^2 - x^2}{a^2 - e^2x^2}, \quad \cos.^2\theta = \frac{x^2(1-e^2)}{a^2 - e^2x^2}:$$

$$\therefore \qquad \overline{OP}^2 = \overline{OT}^2 \times \cos.^2\theta = a^2\frac{a^2(1-e^2)}{a^2 - e^2x^2},$$

or

$$OP = p = a\sqrt{\frac{a^2(1-e^2)}{a^2 - e^2x^2}} = a\sqrt{1 - e^2\sin.^2\theta}.$$

Therefore

$$CM + MP = s + u = a\int\sqrt{1 - e^2\sin.^2\theta}\, d\theta.$$

It is here supposed that the integral takes its origin at C, the vertex of the transverse axis. Now, if the point A be so taken that the angle $BOA = \theta$, it has been shown (Art. 240) that

$$\text{Arc } BA = a\int\sqrt{1 - e^2\sin.^2\theta}\, d\theta:$$

$$\therefore \qquad CM + MP = BA.$$

Also we have

$$MP = -\frac{dp}{d\theta} = \frac{ae^2\sin.\theta\cos.\theta}{\sqrt{1 - e^2\sin.^2\theta}};$$

and, x being the abscissa of the point M,

$$x = p\cos.\theta - \frac{dp}{d\theta}\sin.\theta$$

$$= a(1 - e^2\sin.^2\theta)^{\frac{1}{2}}\cos.\theta + \frac{ae^2\sin.^2\theta\cos.\theta}{(1 - e^2\sin.^2\theta)^{\frac{1}{2}}} = \frac{a\cos.\theta}{(1 - e^2\sin.^2\theta)^{\frac{1}{2}}}.$$

Therefore $MP = e^2 x \sin.\theta$; and, x' being the abscissa of A, we have $x' = a\sin.\theta$: $\therefore MP = \frac{e^2 xx'}{a}$, and hence

$$BA - CM = MP = \frac{e^2}{a}xx',$$

a result known as Fagnani's Theorem.

From the values of x and x', we get

$$x^2 = \frac{a^2 - a^2\sin.^2\theta}{1 - e^2\sin.^2\theta} = \frac{a^2 - x'^2}{1 - \frac{e^2 x'^2}{a^2}};$$

which gives

$$e^2 x^2 x'^2 - a^2(x^2 + x'^2) + a^4 = 0,$$

an equation which is symmetrical with respect to x and x': hence, if we have

$$BA - CM = \frac{e^2}{a}xx',$$

we also have

$$BM - CA = \frac{e^2}{a}xx'.$$

253. *Curves of Double Curvature.* A curve of double curvature is one, three of the consecutive elements of which do not lie in the same plane. Such a curve must be referred to three co-ordinate axes, and requires for its expression two equations which represent the projections of the curve on two of the co-ordinate planes.

Let the equations of the curve be

$$y = f(x) \quad (1), \quad z = \varphi(x) \quad (2);$$

(1) being the equation of the projection on the plane (x, y), and (2) the equation of the projection on the plane (x, z). If x, y, z, are the co-ordinates of a point of the curve, and

$x + \Delta x$, $y + \Delta y$, $z + \Delta z$, the co-ordinates of an adjacent point, then, by the principles of solid geometry, the length of the chord connecting these points is

$$\left\{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right\}^{\frac{1}{2}}.$$

Then, if s is the length of an arc of the curve estimated from a fixed point up to the point (x, y, z), that of the arc from the same fixed point up to the point $(x + \Delta x,\ y + \Delta y,\ z + \Delta z)$ will be expressed by $s + \Delta s$. We shall assume

$$\lim. \frac{\Delta s}{\left\{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right\}^{\frac{1}{2}}}$$

$$= \lim. \frac{\frac{\Delta s}{\Delta x}}{\left\{1 + \left(\frac{\Delta y}{\Delta x}\right)^2 + \left(\frac{\Delta z}{\Delta x}\right)^2\right\}^{\frac{1}{2}}} = 1,$$

and therefore

$$\frac{ds}{dx} = \left\{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right\}^{\frac{1}{2}}:$$

$$\therefore \qquad s = \int \left\{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right\}^{\frac{1}{2}} dx.$$

The two equations of the curve enable us to express $\frac{dy}{dx}$, $\frac{dz}{dx}$, in terms of x; and, by integrating, s will then be known in terms of x.

Any one of the three variables may be taken as independent; and the above formula may be changed into

$$s = \int \left\{1 + \left(\frac{dx}{dy}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{1}{2}} dy,$$

or

$$s = \int \left\{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2\right\}^{\frac{1}{2}} dz.$$

When x, y, and z are each a known function of an auxiliary variable, t, as may be the case, then

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}},\quad \frac{dz}{dx}=\frac{\frac{dz}{dt}}{\frac{dx}{dt}};$$

and we may have

$$s=\int\left\{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right\}^{\frac{1}{2}}dx,$$

or

$$s=\int\left\{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right\}^{\frac{1}{2}}\frac{dx}{dt}\,dt,$$

or

$$s=\int\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}^{\frac{1}{2}}dt.$$

254. To convert the formulæ of the last article into polar formulæ, take the pole at the origin of co-ordinates, and denote by θ the angle that the radius vector makes with the axis of z, and by φ the angle that its projection on the plane (x, y) makes with the axis of x; then we have the relations

$$x=r\sin.\theta\cos.\varphi,\quad y=r\sin.\theta\sin.\varphi,\quad z=r\cos.\theta.$$

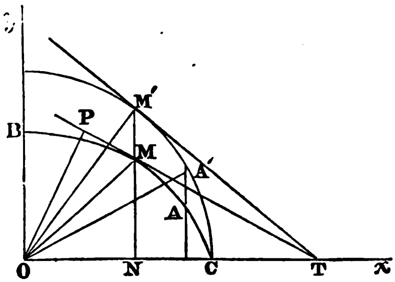
These three equations, together with the two equations of the curve, make five between which we may conceive r and φ to be eliminated, leaving three equations between x, y, z, and θ: hence, x, y, and z may be regarded as known functions of θ.

Therefore

$$\frac{dx}{d\theta}=\sin.\theta\cos.\varphi\frac{dr}{d\theta}-r\sin.\theta\sin.\varphi\frac{d\varphi}{d\theta}+r\cos.\theta\cos.\varphi,$$

$$\frac{dy}{d\theta}\quad\sin.\theta\sin.\varphi\frac{dr}{d\theta}+r\sin.\theta\cos.\varphi\frac{d\varphi}{d\theta}+r\cos.\theta\sin.\varphi,$$

$$\frac{dz}{d\theta}=\cos.\theta\frac{dr}{d\theta}-r\sin.\theta:$$



$$\therefore \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2 \sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2 + r^2, \text{ and}$$

$$s = \int \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 + r^2 \sin.^2\theta \left(\frac{d\varphi}{d\theta}\right)^2 \right\}^{\frac{1}{2}} d\theta;$$

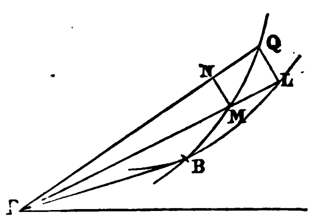
which, by changing the independent variable, may become

$$s = \int \left\{ r^2 \left(\frac{d\theta}{dr}\right)^2 + 1 + r^2 \sin.^2\theta \left(\frac{d\varphi}{dr}\right)^2 \right\}^{\frac{1}{2}} dr,$$

or

$$s = \int \left\{ r^2 \left(\frac{d\theta}{d\varphi}\right)^2 + \left(\frac{dr}{d\varphi}\right)^2 + r^2 \sin.^2\theta \right\}^{\frac{1}{2}} d\varphi.$$

255. *Polar Formula for Plane Areas.* In the curve BMQ of which the polar equation is $r = \varphi(\theta)$, let r, θ, be the co-ordinates of the point M, and denote by A the area bounded by the curve, the radius vector PB drawn to the fixed point B, and the radius vector PM. Then (Art. 165)

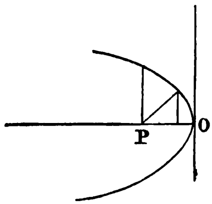


$$\frac{dA}{d\theta} = \frac{1}{2} \left\{ \varphi(\theta) \right\}^2 : \therefore A = \frac{1}{2} \int \left\{ \varphi(\theta) \right\}^2 d\theta.$$

Let $\psi(\theta)$ be the function having $\varphi(\theta)$ for its differential co-efficient; then $A = \psi(\theta) + C$: and if A_1, A_2, denote the areas corresponding to the values θ_1, θ_2, of the vectorial angle, we have

$$A_1 = \psi(\theta_1) + C, \quad A_2 = \psi(\theta_2) + C;$$

$$\therefore \quad A_2 - A_1 = \psi(\theta_2) - \psi(\theta_1) = \frac{1}{2} \int_{\theta_1}^{\theta_2} \left\{ \varphi(\theta) \right\}^2 d\theta.$$



EXAMPLE 1. For the parabola, when the pole is at the focus, and the variable angle, measured from the axis, begins at the vertex, we have $x = p - r\cos.\theta$, $y = r\sin.\theta$; from which, and the equation $y^2 = 2px$ of the curve, we get

$$r = \frac{p}{1+\cos.\theta} = \frac{1}{2}\,\frac{p}{\cos.^2\frac{\theta}{2}}: \quad \therefore A = \frac{p^2}{8}\int\frac{d\theta}{\cos.^4\frac{\theta}{2}}$$

$$= \frac{p^2}{8}\int\left(1+\tan.^2\frac{\theta}{2}\right)\sec.^2\frac{\theta}{2}\,d\theta = \frac{p^2}{4}\tan.\frac{\theta}{2} + \frac{p^2}{12}\tan.^3\frac{\theta}{2} + C;$$

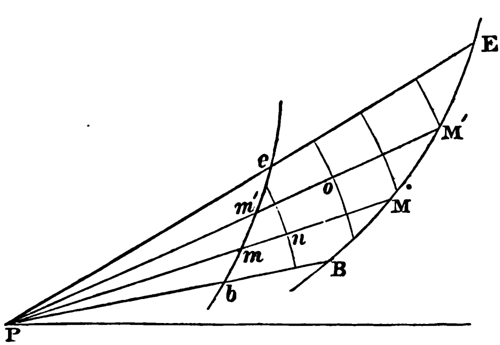
$$\therefore A_2 - A_1 = \frac{p^2}{4}\left(\tan.\frac{\theta_2}{2} - \tan.\frac{\theta_1}{2}\right) + \frac{p^2}{12}\left(\tan.^3\frac{\theta_2}{2} - \tan.^3\frac{\theta_1}{2}\right).$$

Making $\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$, we have for the area $\frac{p^2}{4} + \frac{p^2}{12}$, or $\frac{p^2}{3}$.

Ex. 2. The equation of the logarithmic spiral being $r = be^{\frac{\theta}{c}}$, we find

$$A = \frac{1}{2}\int b^2 e^{\frac{2\theta}{c}}\,d\theta = \frac{b^2c}{4}e^{\frac{2\theta}{c}} + C,$$

$$A_2 - A_1 = \frac{1}{2}\int_{\theta}^{\theta_2} b^2 e^{\frac{2\theta}{c}}\,d\theta = \frac{b^2c}{4}\left(e^{\frac{2\theta_2}{c}} - e^{\frac{2\theta_1}{c}}\right) = \frac{c}{4}\left(r_2^2 - r_1^2\right).$$



256. A polar formula involving double integration may also be constructed for plane areas. Suppose the area included between the curves *BME*, *bme*, and the

radii vectores PB, PE, is required. Divide the area up into curvilinear quadrilaterals by drawing a series of radii vectores, and describing a series of circles with the pole as a centre.

Let no be one of these quadrilaterals, and denote the co-ordinates of n by r, θ; and of o by $r + \Delta r$, $\theta + \Delta\theta$. Now, the area no is the difference between two circular sectors; and the accurate expression for this difference is $r\Delta r\Delta\theta + \frac{1}{2}(\Delta r)^2\Delta\theta$, the ratio of the second term of which to the first is

$$\frac{\frac{1}{2}(\Delta r)^2\Delta\theta}{r\Delta r\Delta\theta} = \frac{\Delta r}{2r}.$$

This ratio diminishes as Δr diminishes, and vanishes when $\Delta r = 0$: therefore we may take $r\Delta r\Delta\theta$ as the expression for the elementary area, since, in comparison with it, the neglected term $\frac{1}{2}(\Delta r)^2\Delta\theta$ ultimately vanishes.

257. In the last article, it was shown that $r\Delta r\Delta\theta$ might be taken as the expression for the polar element of a plane area. If we suppose this area to be the section of a solid by the plane (x, y), the column perpendicular to this plane, standing on the element $r\Delta r\Delta\theta$ as a base, may be regarded as an element of the solid. The volume of this column is measured by $zr\Delta r\Delta\theta$; and therefore, for the volume V of the solid. we have $V = \int\int zrdrd\theta$.

The value of z as a function of r and θ will be given by the equation of the surface bounding the solid.

Example. Required the measure of the volume bounded by the plane (x, y), and the surfaces having

$$x^2 + y^2 - az = 0 \quad (1), \quad x^2 + y^2 - 2bx = 0 \quad (2),$$

for their respective equations.

Denoting the polar co-ordinates of a point in the plane (x, y) by r and θ, θ being measured from the axis of x, we have

$$x = r\cos.\theta,\ \ y = r\sin.\theta \quad (3);$$

therefore $x^2 + y^2 = r^2$, which, combined with (1), gives

$$r^2 = az: \ \therefore\ z = \frac{r^2}{a}.$$

From (2) and (3) we find $r = 2b\cos.\theta$: hence, for this example, we have

$$V = \int\int zrdrd\theta = \int\int \frac{r^3}{a}\,drd\theta.$$

To embrace the entire volume comprised between the surfaces indicated, the integral must first be taken, with respect to r, between the limits $r = 0$, $r = 2b\cos.\theta$, since θ is assumed as the independent variable; and then the integral of the result must be taken between the limits $\theta = \frac{\pi}{2}$, $\theta = -\frac{\pi}{2}$. Thus

$$V = \int_0^{2b\cos.\theta}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^3}{a}\,drd\theta = \frac{4b^4}{a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos.^4\theta d\theta$$

$$= \frac{8b^4}{a}\int_0^{\frac{\pi}{2}} \cos.^4\theta\, d\theta = \frac{3}{2}\,\frac{b^4}{a}\,\pi \quad (\text{Art. } 221).$$

258. Suppose the polar element $r\Delta r\Delta\theta$ of a plane area to revolve through the angle 2π, around the fixed line from which the angle θ is estimated. A solid ring will thus be generated, the measure of which is $2\pi r\sin.\theta r\Delta r\Delta\theta$; since, in this revolution, the point whose polar co-ordinates are r, θ,

will describe a circumference having $r\sin.\theta$ for its radius. Denote by φ the angle which the plane of the generating element in any position makes with its initial position; then $\varphi + \Delta\varphi$ will be the angle which the element in its consecutive position makes with the initial plane. That part of the whole solid ring which is included between the generating element in these two positions is measured by

$$(\varphi + \Delta\varphi)\,r^2\sin.\theta\Delta r\Delta\theta - \varphi r^2\sin.\theta\Delta r\Delta\theta = r^2\sin.\theta\Delta r\Delta\theta\Delta\varphi.$$

This may be assumed as the expression, in terms of polar co-ordinates, for an element of the solid: hence, for the volume V of the whole solid, we have

$$V = \int\int\int r^2\sin.\theta dr d\theta d\varphi,$$

in which the limits of integration must be so determined from imposed conditions, that the integral may embrace the entire solid to be found.

EXAMPLE. Required the volume of a tri-rectangular pyramid in a sphere. Integrating the above formula, with respect to r, between the limits $r = 0$, $r = a$, a being the radius of the sphere, we find

$$V = \int_0^a\int\int r^2\sin.\theta dr d\theta dx = \int\int\frac{a}{3}\sin.\theta d\theta d\varphi.$$

Now, $a^2\sin.\theta\Delta\theta\Delta\varphi$ is an element of the spherical surface; and $\frac{a^3}{3}\sin.\theta\Delta\theta\Delta\varphi$ is therefore the expression for an elementary spherical pyramid having $a^2\sin.\theta\Delta\theta\Delta\varphi$ for its base. By this first integration, therefore, the element of the volume has changed from an element of the solid ring, generated by the revolution of $r\Delta r\Delta\theta$, to an elementary spherical pyramid.

Integrating next, with respect to θ, between the limits

$$\theta = 0,\ \theta = \frac{\pi}{2},$$

we have

$$V = \int \frac{a^3}{3} d\varphi;$$

since

$$\int \sin.\theta d\theta = -\cos.\theta: \quad \therefore \int_{\frac{\pi}{2}}^{0} \sin.\theta d\theta = 1.$$

By this second integration, the elementary volume has become a semi-ungula, or a spherical pyramid, having a bi-rectangular triangle for its base; the vertical angle of the triangle being $\Delta\varphi$.

We finally integrate, with respect to φ, from $\varphi = 0$ to $\varphi = \frac{\pi}{2}$, and get for our result

$$V = \frac{\pi a^3}{6}.$$

SECTION IX.

DIFFERENTIATION AND INTEGRATION UNDER THE SIGN $\int$. — EULERIAN INTEGRALS. — DETERMINATION OF DEFINITE INTEGRALS BY DIFFERENTIATION, AND BY INTEGRATION UNDER THE SIGN $\int$.

259. WHATEVER function of x, $f(x)$ may be, there exists another function, $\varphi(x)$, of x, such that $\varphi'(x) = f(x)$; and therefore $\int f(x)dx = \varphi(x) + C$ (Art. 191), C being an arbitrary constant.

Denoting by u the integral of $f(x)dx$, taken between the limits a and b, we have

$$u = \int_a^b f(x)dx = \varphi(b) - \varphi(a).$$

The definite integral u is independent of x, but is a function of the limits a and b; and its differential co-efficient with respect to either of these limits may be obtained without effecting the integration. For, since

$$u = \varphi(b) - \varphi(a),$$

we have

$$\frac{du}{da} = -\varphi'(a), \quad \frac{du}{db} = \varphi'(b);$$

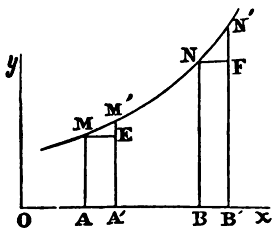
and, because $\varphi'(x) = f(x)$,

$$\frac{du}{da} = -f(a), \quad \frac{du}{db} = f(b):$$

$$\therefore \quad du = f(b)db - f(a)da.$$

ILLUSTRATION.

Let $y = f(x)$ be the equation of the curve MN referred to the rectangular axes Ox, Oy. If a and b are the abscissæ of the points M and N, $u = \int_a^b f(x)\,dx$ will represent the area $AMNB$. Give to a and b the increments



$$AA' = \Delta a,\ BB' = \Delta b;$$

then

$$\frac{\Delta u}{\Delta a} = \frac{AMM'A'}{AA'},\ \frac{\Delta u}{\Delta b} = \frac{BNN'B'}{BB'}.$$

The definite area $AMNB$ is obviously a decreasing function of the first limit a, and an increasing function of the second limit b: therefore

$$\lim.\ \frac{\Delta u}{\Delta a} = \lim.\ \frac{AMM'A'}{AA'} = -f(a),$$

$$\lim.\ \frac{\Delta u}{\Delta b} = \lim.\ \frac{BNN'B'}{BB'} = f(b).$$

Regarding the areas $AMM'A'$, $BNN'B'$, as elementary, we see that the total increment of the area $AMNB$ is the difference of the increments that it receives at the limits.

260. Suppose $f(x)$ to contain a quantity, t, independent of x, and that the differential co-efficient of $\int_a^b f(x)\,dx$ with respect to t is reouired. Replacing $f(x)$ by $f(x, t)$, we have

$$u = \int_a^b f(x, t)\,dx.$$

If the limits a and b are independent of t we have by giv-

ing to t the increment Δt (Δu being the corresponding increment of u),

$$\Delta u = \int_a^b f(x, t + \Delta t)\,dx - \int_a^b f(x, t)\,dx$$

$$= \int_a^b \Big(f(x, t + \Delta t) - f(x, t)\Big)\,dx:$$

$$\therefore \qquad \frac{\Delta u}{\Delta t} = \int_a^b \frac{f(x, t + \Delta t) - f(x, t)}{\Delta t}\,dx.$$

Now, by Art. 15, we may write

$$\frac{f(x, t + \Delta t) - f(x, t)}{\Delta t} = \frac{df(x, t)}{dt} + \gamma,$$

in which γ is a quantity that vanishes when Δt vanishes. Denoting by γ' the greatest of the values of γ, we have, generally,

$$\int_a^b \gamma\,dx < (b - a)\gamma';$$

and, when neither a nor b is infinite, $(b - a)\gamma'$, and therefore $\int_a^b \gamma\,dx$, will ultimately vanish:

$$\therefore \qquad \lim. \frac{\Delta u}{\Delta t} = \frac{du}{dt} = \int_a^b \frac{df(x, t)}{dt}\,dx.$$

APPLICATION.

Resuming the formula

$$\frac{du}{dt} = \frac{d}{dt}\int_a^b f(x, t) = \int_a^b \frac{df(x, t)}{dt}\,dx \quad (1)$$

just established, suppose $\varphi(x, t)$ to be the function of which $f(x, t)$ is the differential co-efficient with respect to x, and $\psi(x, t)$ to be the function of which $\dfrac{df(x, t)}{dt}$ is the differential co-efficient with respect to x; then (1) becomes

$$\frac{d\varphi(b, t)}{dt} - \frac{d\varphi(a, t)}{dt} = \psi(b, t) - \psi(a, t) \quad (2).$$

If $f(x, t)$ and a are both independent of b, (2) may be written

$$\frac{d\varphi(b,t)}{dt} + C = \psi(b,t) \quad (3);$$

C denoting the sum of the terms which are independent of b. Since we may give to b in (3) any value we please, replace b by x; then (3) becomes

$$\psi(x,t) = \frac{d\varphi(x,t)}{dt} + C \quad (4).$$

Dropping the constant C, which may be restored when necessary, and putting for the other terms of (4) their equivalents, we have

$$\int \frac{df(x,t)}{dt}\,dx = \frac{d}{dt}\int f(x,t)\,dx.$$

Example. Let $f(x,t) = \dfrac{1}{1+t^2x^2}$, then $f(x,t)dx = \dfrac{dx}{1+t^2x^2}$:

$$\therefore \qquad \int f(x,t)\,dx = \int \frac{dx}{1+t^2x^2} = \frac{1}{t}\tan.^{-1}tx,$$

and $\dfrac{d}{dt}\displaystyle\int f(x,t)\,dx = \dfrac{d}{dt}\left(\dfrac{1}{t}\tan.^{-1}tx\right) = \displaystyle\int \dfrac{df(x,t)}{dt}\,dx$

$$= \int \frac{d}{dt}\left(\frac{1}{1+t^2x^2}\right)dx = -\int \frac{2tx^2}{(1+t^2x^2)^2}\,dx.$$

Thus, having the value of $\displaystyle\int \frac{dx}{1+t^2x^2}$, we find, by differentiation, that of the more complex integral $\displaystyle\int \frac{2tx^2}{1+t^2x^2}\,dx$.

261. When, in the integral $u = \displaystyle\int_a^b f(x,t)\,dx$, both a and b are functions of t, then $\dfrac{du}{dt}$ will consist of three terms; since in this case, to obtain the total differential of u, we must differentiate it with respect to t, and also with respect to both

a and b regarded as functions of t, and take the sum of the results. Thus we should have

$$\frac{du}{dt}=\int_a^b \frac{df(x,t)}{dt}dx+\frac{du}{db}\frac{db}{dt}+\frac{du}{da}\frac{da}{dt}$$

$$=\int_a^b \frac{df(x,t)}{dt}dx+f(b,t)\frac{db}{dt}-f(a,t)\frac{da}{dt} \quad \text{(Art. 259).}$$

Under the above suppositions, the second and higher differential co-efficients of u with respect to t may be found. Thus, by differentiating each of the terms of the last formula with respect to t, we get

$$\frac{d^2u}{dt^2}=\int_a^b \frac{d^2f(x,t)}{dt^2}dx$$

$$+f(b,t)\frac{d^2b}{dt^2}+\frac{df(b,t)}{db}\left(\frac{db}{dt}\right)^2+2\frac{df(b,t)}{dt}\frac{db}{dt}$$

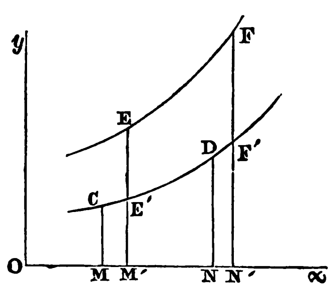
$$-f(a,t)\frac{d^2a}{dt^2}-\frac{df(a,t)}{da}\left(\frac{da}{dt}\right)^2-2\frac{df(a,t)}{dt}\frac{da}{dt}.$$

ILLUSTRATION.

Let $y=f(x,t)$ be the equation of the curve CD referred to the rectangular axes Ox, Oy, and $y=f(x,t+\Delta t)$ that of the curve EF. Put

$$OM=a, \quad ON=b,$$

$$MM'=\Delta a, \quad NN'=\Delta b.$$



Then $u=\int_a^b f(x,t)\,dx$ denotes the area $MNDC$, and $u+\Delta u$ the area $M'N'FE$:

$$\therefore \qquad \Delta u=EE'F'F+DNN'F'-MM'E'C,$$

$$\frac{\Delta u}{\Delta t}=\frac{EE'F'F}{\Delta t}+\frac{DNN'F'}{\Delta t}-\frac{MM'E'C}{\Delta t}.$$

It is plain that the first term in this value of $\frac{\Delta u}{\Delta t}$ is the ratio of Δt to the increment of the area due to the change from the curve CD to the curve EF. The limit of this ratio is the limit of $\int_a^b \frac{f(x, t+\Delta t) - f(x, t)}{\Delta t} dx$. So, also, the limit of the second term is the limit of $f(b, t)\frac{\Delta b}{\Delta t}$, and the limit of the third term is the limit of $f(a, t)\frac{\Delta a}{\Delta t}$: hence

$$\frac{du}{dt} = \int_a^b \frac{df(x, t)}{dt} dx + f(b, t)\frac{db}{dt} - f(a, t)\frac{da}{dt},$$

which agrees with first formula established in this article.

262. An indefinite integral may also be differentiated with respect to a variable contained in the function under the sign of integration which is independent of the variable to which the integration refers.

Let the integral be $u = \int f(x, t)dx$, t being independent of x: then, without impairing the generality of this integral, we may write

$$u = \int_a^x f(x, t)dx + \psi(t);$$

$\psi(t)$ being an arbitrary function of t. Differentiating with respect to t, t not depending on a, we have (Art. 260)

$$\frac{du}{dt} = \int_a^x \frac{df(x, t)}{dt} dx + \psi'(t):$$

but, since $\psi'(t)$ is a constant with respect to x, it may be included in the constant of the integral $\int \frac{df(x, t)}{dt} dx$; and hence the last equation may be written

$$\frac{du}{dt} = \int \frac{df(x, t)}{dt} dx,$$

and we have only to differentiate the function under the sign $\int$ with respect to t.

263. *Integration under the Sign of Integration.* Taking the definite integral $\int_a^b f(x,y)\,dx$ as the differential co-efficient of y, and integrating, we have

$$\int dy \int_a^b f(x,y)\,dx$$

for our result; and it is proposed to prove that this result is the same in whichever order with respect to x and y the integrations are performed; that is, we shall have

$$\int dy \int_a^b f(x,y)\,dx = \int_a^b dx \int f(x,y)\,dy.$$

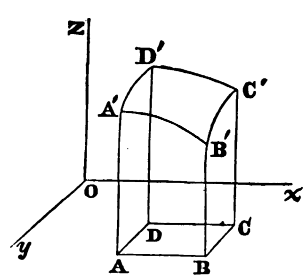
For
$$\frac{d}{dy}\int_a^b dx \int f(x,y)\,dy = \int_a^b dx\,\frac{d\int f(x,y)\,dy}{dy}$$
$$= \int_a^b f(x,y)\,dx.$$

Integrating the two members of this equation with respect to y, we get

$$\int_a^b dx \int f(x,y)\,dy = \int dy \int_a^b f(x,y)\,dx;$$

and, if the limiting values of y are c and d, we shall have

$$\int_a^b dx \int_c^d f(x,y)\,dy$$
$$= \int_c^d dy \int_a^b f(x,y)\,dx.$$



The figure gives the geometrical interpretation of this formula. Either member represents the volume AC' included between the plane (x,y), the surface $A'B'C'D'$ having $z = f(x,y)$ for its

equation, and the planes whose equations are $x=a$, $x=b$, $y=c$, $y=d$.

EXAMPLE 1. Find the form of the function $\varphi(x)$ such that the area included between the curve $y=\varphi(x)$, the axis of x, and the ordinates $y=0$, $y=\varphi(a)$, shall bear a constant ratio, n, to the rectangle contained by the latter ordinate and the corresponding abscissa.

By the conditions, we must have

$$\int_0^a \varphi(x)dx = \frac{a\varphi(a)}{n};$$

and, since this is to hold for all values of a, we may differentiate with respect to a: hence

$$\varphi(a) = \frac{\varphi(a)}{n} + a\frac{\varphi'(a)}{n}:$$

$$\therefore \quad \frac{\varphi'(a)}{\varphi(a)} = \frac{n-1}{a};$$

and by integration

$$l\varphi(a) = (n-1)\,la + C.$$

Passing from logarithms to numbers,

$$\varphi(a) = Ca^{n-1}: \therefore \varphi(x) = Cx^{n-1};$$

and the equation of the curve is $y = Cx^{n-1}$.

Ex. 2. Determine such a form for $\varphi(x)$ that the integral $u = \int_0^a \frac{\varphi(x)dx}{\sqrt{(a-x)}}$ shall be independent of a.

Put $x = az$; then, since the limits $x=0$, $x=a$, correspond to $z=0$, $z=1$,

$$u = \int_0^a \frac{\varphi(x)dx}{\sqrt{(a-x)}} = \int_0^1 \frac{\sqrt{a}\varphi(az)dz}{\sqrt{(1-z)}}.$$

By condition, u is to be independent of a: therefore the differential co-efficient of u, with respect to a, must be zero. But

$$\frac{du}{da} = \int_0^1 \frac{\frac{\varphi(az)}{2\sqrt{a}} + z\sqrt{a}\varphi'(az)}{\sqrt{(1-z)}} = \int_0^a \frac{\varphi(x) + 2x\varphi'(x)}{2a\sqrt{(a-x)}} dx;$$

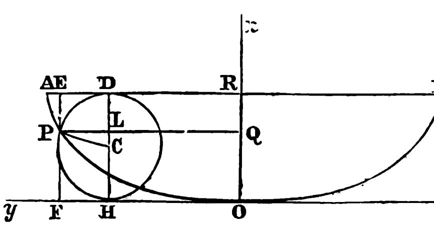
and, since this last integral is to be zero for all values of a, we must have

$$\varphi(x) + 2x\varphi'(x) = 0: \quad \therefore \frac{\varphi'(x)}{\varphi(x)} = -\frac{1}{2x}.$$

Therefore $$l\varphi(x) = -\frac{1}{2}lx + C,$$

or $$\varphi(x) = \frac{C}{\sqrt{x}}.$$

Let AOB be a cycloid, with its vertex downwards; and let it be referred to the axis Ox, and the tangent through its vertex, as co-ordinate axes.



Then, denoting the angle DCP by θ, we have for the co-ordinates $OF = y$, $OQ = x$ of the point P,

$$x = FP = HL = a + a\cos.\theta,$$
$$y = OF = AR - AE = AR - AD + ED$$
$$= a\pi - a\theta + a\sin.\theta.$$

Put $\theta = \pi - \varphi$, then these values of x and y become

$$x = a - a\cos.\varphi \quad (1), \quad y = a\varphi + a\sin.\varphi \quad (2).$$

From (1) we find

$$\varphi = \cos.^{-1}\frac{a-x}{a}, \quad \sin.\varphi = \frac{1}{a}\sqrt{2ax - x^2};$$

and thus (2) becomes

$$y = a \cos.^{-1} \frac{a - x}{a} + \sqrt{2ax - x^2},$$

which is the equation of the cycloid. By differentiation, we get

$$\frac{dy}{dx} = \sqrt{\frac{2a - x}{a}}; \quad \therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{2a}{x}};$$

and, by integration, $s = \sqrt{8ax}$. We therefore conclude that $\varphi(x)$, in Ex. 2, is the expression for the arc of a cycloid estimated from the vertex.

This example is the solution of the problem in mechanics for finding the curve down which bodies, starting from different points, will fall in equal times.

264. ***The Eulerian Integral of the First Species*** is an integral of the form

$$\int_0^1 x^{p-1}(1 - x)^{q-1} \, dx,$$

in which p and q are positive numbers. This is denoted by $B(p, q)$.

The Eulerian Integral of the Second Species is of the form

$$\int_0^\infty e^{-x} x^{n-1} \, dx,$$

and is denoted by $\Gamma(n)$.

The first species may be put under the two forms

$$\int_0^\infty \frac{y^{p-1} \, dy}{(1 + y)^{p+q}}, \quad 2\int_0^{\frac{\pi}{2}} \sin.^{2p-1} \theta \cos.^{2q-1} \theta d\theta,$$

by making $x = \dfrac{y}{1 + y}$ for the first form, and $x = \sin.^2 \theta$ for the second.

The integral of the first species is a symmetrical function of p and q; for, making $x = 1 - y$, we have

$$B(p, q) = \int_1^0 (1-y)^{p-1} y^{q-1} dy = B(q, p):$$

$$\therefore \quad B(p, q) = B(q, p).$$

265. Integrating by parts, we have

$$\int x^p (1-x)^{q-1} dx$$

$$= -\frac{x^p (1-x)^q}{q} + \frac{p}{q}\int x^{p-1} (1-x)^{q-1} (1-x)\, dx$$

$$= -\frac{x^p (1-x)^q}{q} + \frac{p}{q}\int x^{p-1}(1-x)^{q-1} dx - \frac{p}{q}\int x^p (1-x)^{q-1} dx.$$

Therefore, taking 1 and 0 for the limits, we have

$$B(p+1, q) = \frac{p}{q} B(p, q) - \frac{p}{q} B(p+1, q):$$

$$\therefore \quad B(p+1, q) = \frac{p}{p+q} B(p, q).$$

In like manner,

$$B(p, q+1) = \frac{q}{p+q} B(p, q).$$

In the integral of the first species, therefore, each of the exponents p and q may be diminished by unity.

266. In the Eulerian integral of the second species

$$\int_0^\infty e^{-x} x^{n-1} dx,$$

n must be positive, otherwise the integral would be infinite. For if n be negative, and equal to $-p$, we should have

$$\int_0^\infty e^{-x} x^{n-1} dx = \int_0^\infty \frac{1}{e^x x^{p+1}} dx:$$

and it is plain, that, when $x = \infty$, the differential co-efficient is zero, and therefore the integral is zero; and, when $x = 0$, the

differential co-efficient is infinite, and therefore the integral is infinite.

The integral $\Gamma(n+1)$ may be made to depend on $\Gamma(n)$. For, integrating by parts, we have

$$\int e^{-x}x^n dx = -e^{-x}x^n + n\int e^{-x}x^{n-1}dx.$$

But $e^{-x}x^n$ reduces to zero both when $x=0$ and when $x=\infty$ (Ex. 3, Art. 103): therefore

$$\int_0^\infty e^{-x}x^n dx = n\int_0^\infty e^{-x}x^{n-1}dx,$$

or
$$\Gamma(n+1) = n\Gamma(n).$$

In like manner,

$$\Gamma(n) = (n-1)\Gamma(n-1),\ \Gamma(n-1) = (n-2)\Gamma(n-2);$$

and, if n is entire, we shall have, finally,

$$\Gamma(2) = \Gamma(1),\ \Gamma(1) = \int_0^\infty e^{-x}dx = 1.$$

Therefore, when n is an entire and positive number, we shall have

$$\Gamma(n) = 1.2.3\ldots(n-1);$$

and, if n is a fraction greater than 1, then the formula

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

enables us to reduce the integral $\Gamma(n)$ to that of $\Gamma(\mu)$, μ denoting a number less than 1. Hence, to compute the value $\Gamma(n)$, it is sufficient to know the values of this function for values of n between 0 and 1.

267. By putting $e^{-x}=y$, the integral $\Gamma(n)$ may be made to take another form. Thus, from $e^{-x}=y$, we get

$$x = l\frac{1}{y},\ dx = -\frac{dy}{y}:$$

$$\therefore\ \int_0^\infty e^{-x}x^{n-1}dx = -\int_1^0\left(l\frac{1}{y}\right)^{n-1}dy = \int_0^1\left(l\frac{1}{y}\right)^{n-1}dy,$$

or
$$\Gamma(n) = \int_0^1\left(l\frac{1}{y}\right)^{n-1}dy.$$

268. *Relations between the two Eulerian Integrals.* Assume the double integral

$$\int_0^\infty \int_0^\infty x^{p+q-1}\, y^{p-1}\, e^{-(1+y)x}\, dy dx,$$

and integrate with respect to x: it thus becomes

$$\Gamma(p+q)\int_0^\infty \frac{y^{p-1}dy}{(1+y)^{p+q}} \quad \text{(Art. 266).}$$

Integrating the same double integral with respect to y, it becomes

$$\Gamma(p)\int_0^\infty \frac{e^{-x}x^{p+q-1}dx}{x^p} = \Gamma(p)\int_0^\infty e^{-x}x^{q-1}\,dx = \Gamma(p)\Gamma(q):$$

therefore

$$\Gamma(p+q)\int_0^\infty \frac{y^{p-1}dy}{(1+y)^{p+q}} = \Gamma(p)\Gamma(q):$$

$$\therefore \qquad \int_0^\infty \frac{y^{p-1}dy}{(1+y)^{p+q}} = B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)};$$

that is,

$$\int_0^1 x^{p-1}(1-x)^{q-1}dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Putting $\frac{z}{a}$ for x in the first member of this last equation, we have

$$\int_0^a \frac{z^{p-1}}{a^{p-1}}\,\frac{(a-z)^{q-1}}{a^{q-1}}\,\frac{dz}{a} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

or $$\int_0^a z^{p-1}(a-z)^{q-1}dz = a^{p+q-1}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

269. The last formula in the preceding article is a particular case of a more general formula by which may be expressed, in terms of Γ functions, the multiple integral

$$\iiint \ldots x^{p-1}y^{q-1}z^{r-1}\ldots(a-x-y-z\ldots)^{s-1}dxdydz\ldots$$

extended to all positive values of x, y, $z\ldots$, which satisfy the condition $x+y+z\ldots < a$.

Limiting ourselves to three variables, let

$$A=\int_0^a x^{p-1}dx\int_0^{a-x} y^{q-1}dy\int_0^{a-x-y} z^{r-1}(a-x-y-z)^{s-1}dz.$$

Now, by the last article,

$$\int_0^{a-x-y} z^{r-1}(a-x-y-z)^{s-1}=(a-x-y)^{r+s-1}\frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}.$$

Multiplying this by $y^{q-1}dy$, and integrating with respect to y from $y=0$ to $y=a-x$, the result is

$$(a-x)^{q+r+s-1}\frac{\Gamma(q)\Gamma(r+s)}{\Gamma(q+r+s)}\,\frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

$$=(a-x)^{q+r+s-1}\frac{\Gamma(q)\Gamma(r)\Gamma(s)}{\Gamma(q+r+s)};$$

and finally, multiplying this last by $x^{p-1}dx$, and integrating with respect to x from $x=0$ to $x=a$, we have

$$A=a^{p+q+r+s-1}\frac{\Gamma(p)\Gamma(q)\Gamma(r)\Gamma(s)}{\Gamma(p+q+r+s)} \quad (1).$$

In this, making $a=1$, $s=1$, we have

$$\iiint x^{p-1}y^{q-1}z^{r-1}\,dxdydz=\frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(p+q+r+1)};$$

the limits of integration being any positive values of x, y, z, which satisfy the inequality $x+y+z<1$.

Assume $\left(\frac{x}{a}\right)^\alpha=\mu,\ \left(\frac{y}{b}\right)^\beta=\eta,\ \left(\frac{z}{c}\right)^\gamma=\zeta;$

then
$$A=\frac{a^pb^qc^r}{\alpha\beta\gamma}\iiint \mu^{\frac{p}{\alpha}-1}\eta^{\frac{q}{\beta}-1}\zeta^{\frac{r}{\gamma}-1}d\mu d\eta d\zeta,$$

subject to the condition that $\mu+\eta+\zeta<1$: therefore

$$A=\frac{a^pb^qc^r}{\alpha\beta\gamma}\,\frac{\Gamma\left(\frac{p}{\alpha}\right)\Gamma\left(\frac{q}{\beta}\right)\Gamma\left(\frac{r}{\gamma}\right)}{\Gamma\left(\frac{p}{\alpha}+\frac{q}{\beta}+\frac{r}{\gamma}+1\right)} \quad (2).$$

270. By means of Formula 2 of the last article, we can find the volume bounded by the co-ordinate planes, and the surface having for its equation

$$\left(\frac{x}{a}\right)^{\alpha}+\left(\frac{y}{b}\right)^{\beta}+\left(\frac{z}{c}\right)^{\gamma}=1.$$

EXAMPLE. When $\alpha=\beta=\gamma=2$, and $p=q=r=1$, the surface is that of an ellipsoid of which $2a$, $2b$, $2c$, are the axes. Then, by the formula, the volume V of $\frac{1}{8}$ of this ellipsoid will be

$$V=\frac{abc}{8}\frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^{3}}{\Gamma\left(\frac{3}{2}+1\right)}.$$

But $$\Gamma\left(\frac{3}{2}+1\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{3}{4}\Gamma\left(\frac{1}{2}\right)\quad \text{(Art. 266)},$$

and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$; for, let $u=\int_0^\infty e^{-x^2}dx$, then also

$$u=\int_0^\infty e^{-y^2}dy,$$

and $$u^2=\int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy=\int_0^\infty\int_0^\infty e^{-x^2-y^2}dxdy.$$

Now, $\int_0^\infty\int_0^\infty e^{-x^2-y^2}dxdy$ is obviously $\frac{1}{4}$ of the volume the equation of whose surface is $z=e^{-x^2-y^2}$. In terms of polar co-ordinates, the expression for the same part of the volume is

$$\int_0^{\frac{\pi}{2}}\int_0^\infty zrd\theta dr=\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rd\theta dr.$$

But

$$\int e^{-r^2}rdr=-\frac{1}{2}e^{-r^2};\quad \therefore \int_0^\infty e^{-r^2}rdr=\frac{1}{2}:$$

and $\int d\theta = \theta$; $\therefore \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{1}{4}\pi$;

$$\therefore \qquad u = \int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}.$$

Now, $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} x^{-\frac{1}{2}}\,dx$ by definition,

$= 2\int_0^\infty e^{-y^2}\,dy = 2u = \sqrt{\pi}$ by putting $x = y^2$:

therefore, $$V = \frac{abc}{8}\,\frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^3}{\Gamma\left(\frac{3}{2}+1\right)} = \frac{abc}{6}\pi.$$

271. Differentiation under the sign $\int$ enables us to find new integrals from known definite integrals. Thus,

EXAMPLE 1. $$\int_0^\infty \frac{dx}{x^2+a} = \frac{\pi}{2}a^{-\frac{1}{2}}.$$

Differentiating each member of this equation n times with respect to a, we get

$$\int_0^\infty \frac{1.2.3\ldots n}{(x^2+a)^{n+1}}\,dx = \frac{1}{2}\,\frac{3}{2}\,\frac{5}{2}\cdots\frac{2n-1}{2}\,\frac{1}{a^{n+\frac{1}{2}}}\,\frac{\pi}{2};$$

whence $$\int_0^\infty \frac{dx}{(x^2+a)^{n+1}} = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\,\frac{\pi}{2a^{n+\frac{1}{2}}}.$$

Ex. 2. $$\int_0^\infty e^{-ax}\,dx = \frac{1}{a}.$$

After $n-1$ differentiations of the two members of this equation, it becomes

$$\int_0^\infty e^{-ax} x^{n-1}\,dx = 1.2.3\ldots(n-1)a^{-n};$$

that is $$\int_0^\infty e^{-ax} x^{n-1}\,dx = \frac{\Gamma(n)}{a^n} \quad \text{(Art. 266).}$$

The last formula holds good when a is replaced by the imaginary quantity $a+b\sqrt{-1}$, in which a is positive; for

$$\int e^{-(a+b\sqrt{-1})x}\,dx$$

$$= -\frac{e^{-(a+b\sqrt{-1})x}}{a+b\sqrt{-1}} + C$$

$$= -\frac{e^{-ax}(\cos.bx - \sqrt{-1}\sin.bx)}{a+b\sqrt{-1}} + C \text{ (Art. 73)}:$$

therefore

$$\int_0^\infty e^{-(a+b\sqrt{-1})x}\,dx = \frac{1}{a+b\sqrt{-1}};$$

and, by differentiating this equality $n-1$ times with respect to a, we get

$$\int_0^\infty e^{-(a+b\sqrt{-1})x}\,x^{n-1}\,dx = \frac{1.2.3\ldots(n-1)}{(a+b\sqrt{-1})^n}.$$

272. The formula just found leads to other integrals by the separation of real from imaginary quantities.

Assume $a+b\sqrt{-1} = \rho(\cos.\theta + \sqrt{-1}\sin.\theta)$, in which

$$\rho = \sqrt{a^2+b^2},\quad \cos.\theta = \frac{a}{\sqrt{a^2+b^2}},\quad \sin.\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

Then

$$\int_0^\infty e^{-(a+b\sqrt{-1})x}\,x^{n-1}\,dx$$

$$= \int_0^\infty e^{-ax}(\cos.bx - \sqrt{-1}\sin.bx)\,x^{n-1}\,dx$$

and

$$\frac{1.2.3\ldots n-1}{(a+b\sqrt{-1})^n} = \frac{\Gamma(n)}{\rho^n}\,\frac{1}{\cos.n\theta + \sqrt{-1}\sin.n\theta}$$

$$= \frac{\Gamma(n)}{\rho^n}(\cos.n\theta - \sqrt{-1}\sin.n\theta):$$

$$\therefore \int_0^\infty e^{-ax}(\cos. bx - \sqrt{-1}\sin. bx)\, x^{n-1}\, dx$$

$$= \frac{\Gamma(n)}{\rho^n}(\cos. n\theta - \sqrt{-1}\sin. n\theta);$$

an equation which may be separated into the two,

$$\int_0^\infty e^{-ax} x^{n-1} \sin. bx dx = \frac{\Gamma(n)}{\rho^n} \sin. n\theta,$$

$$\int_0^\infty e^{-ax} x^{n-1} \cos. bx dx = \frac{\Gamma(n)}{\rho^n} \cos. n\theta.$$

273. Making $n = 1$ in the last formula of the preceding article, it becomes

$$\int_0^\infty e^{-ax} \cos. bx dx = \frac{a}{a^2+b^2}:$$

therefore, denoting by c a constant less than a, we have

$$\int_c^a da \int_0^\infty e^{-ax} \cos. bx dx = \int_c^a \frac{a da}{a^2+b^2}.$$

But

$$\int_c^a da \int_0^\infty e^{-ax} \cos. bx dx = \int_0^\infty dx \int_c^a e^{-ax} \cos. bx da$$

$$= \int_0^\infty \frac{e^{-cx} - e^{-ax}}{x} \cos. bx dx.$$

Again:

$$\int_c^a \frac{a da}{a^2+b^2} = \frac{1}{2}\, l\, \frac{a^2+b^2}{c^2+b^2};$$

$$\therefore \int_0^\infty \frac{e^{-cx} - e^{-ax}}{x} \cos. bx dx = \frac{1}{2}\, l\, \frac{a^2+b^2}{c^2+b^2}.$$

Making $b = 0$ in the last equation, it becomes

$$\int_0^\infty \frac{e^{-cx} - e^{-ax}}{x} dx = l\,\frac{a}{c};$$

a result that may also be obtained by multiplying both members of the equation

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$$

by da, and integrating the result between the limits a and c.

274. In like manner, from the formula

$$\int_0^\infty e^{-ax}\sin.bx\,dx = \frac{b}{a^2+b^2},$$

we get

$$\int_c^a\int_0^\infty e^{-ax}\sin.bx\,dx = \int_c^a \frac{b\,da}{a^2+b^2}$$

$$= \tan.^{-1}\frac{a}{b} - \tan.^{-1}\frac{c}{b}.$$

But

$$\int_c^a\int_0^\infty e^{-ax}\sin.bx\,dx = \int_0^\infty dx\int_c^a \sin.bx\,e^{-ax}\,da$$

$$= \int_0^\infty \frac{e^{-cx}-e^{-ax}}{x}\sin.bx\,dx:$$

$$\therefore\quad \int_0^\infty \frac{e^{-cx}-e^{-ax}}{x}\sin.bx\,dx = \tan.^{-1}\frac{a}{b} - \tan.^{-1}\frac{c}{b}.$$

In this formula, making $a=\infty$, $c=0$, it reduces to

$$\int_0^\infty \frac{\sin.bx}{x}\,dx = \frac{\pi}{2}$$

when $b \gtrless {}^{0}_{\infty}$; when $b \lessgtr {}^{0}_{-\infty}$, the second member becomes $-\dfrac{\pi}{2}$: from which it is seen that the integral $\displaystyle\int_0^\infty \frac{\sin.bx}{x}\,dx$ changes abruptly from $\dfrac{\pi}{2}$ to $-\dfrac{\pi}{2}$, when b, in passing through zero, changes from positive to negative.

275. The integral $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$ (Ex. Art. 270) leads to $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$; for

$$\int_{-\infty}^\infty e^{-x^2}\,dx = \int_{-\infty}^0 e^{-x^2}\,dx + \int_0^\infty e^{-x^2}\,dx.$$

Now, if we change x into $-x$, we have

$$\int_{-\infty}^0 e^{-x^2}\,dx = \int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}:$$

$$\therefore \qquad \int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}.$$

And generally, if $f(x)$ is a function of the even powers of x, that is, such a function that $f(x) = f(-x)$, then

$$\int_{-\infty}^\infty f(x)\,dx = 2\int_0^\infty f(x)\,dx;$$

for $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx.$$

But $$\int_{-\infty}^0 f(x)\,dx = \int_0^\infty f(-x)\,dx = \int_0 f(x)\,dx:$$

$$\therefore \qquad \int_{-\infty}^\infty f(x)\,dx = 2\int_0^\infty f(x)\,dx.$$

In like manner, it may be shown that if $f(x)$ is a function of the odd powers of x, that is, if $f(-x) = -f(x)$, we should have

$$\int_{-\infty}^\infty f(x)\,dx = 0.$$

276. In the integral $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$, putting $x\sqrt{a}$ for x, we have

$$\int_{-\infty}^\infty e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{\sqrt{a}};$$

which, by n differentiations with respect to a, becomes

$$\int_{-\infty}^\infty e^{-ax^2} x^{2n}\,dx = \sqrt{\pi}\,\frac{1.3.5\ldots(2n-1)}{2^n}\,a^{-(n+\frac{1}{2})}.$$

In this, making $a = 1$, we have

$$\int_{-\infty}^{\infty} e^{-x^2} x^{2n} dx = \sqrt{\pi}\, \frac{1.3.5\ldots(2n-1)}{2^n}.$$

277. Changing x into $x + a$ in the formula

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

of the preceding article, we get

$$\int_{-\infty}^{\infty} e^{-(x+a)^2} dx = \sqrt{\pi};$$

that is,
$$e^{-a^2} \int_{-\infty}^{\infty} e^{-x^2-2ax} dx = \sqrt{\pi}:$$

$$\therefore \quad \int_{-\infty}^{\infty} e^{-x^2-2ax} dx = e^{a^2} \sqrt{\pi}.$$

But $\int_{-\infty}^{\infty} e^{-x^2-2ax} dx = \int_{-\infty}^{0} e^{-x^2-2ax} dx + \int_{0}^{\infty} e^{-x^2-2ax} dx,$

and $\int_{-\infty}^{0} e^{-x^2-2ax} dx = \int_{0}^{\infty} e^{-x^2+2ax} dx$

by changing x into $-x$:

$$\therefore \quad \int_{-\infty}^{\infty} e^{-x^2-2ax} dx = \int_{0}^{\infty} e^{-x^2+2ax} dx + \int_{0}^{\infty} e^{-x^2-2ax} dx$$

$$= \int_{0}^{\infty} e^{-x^2} (e^{2ax} + e^{-2ax}) dx;$$

whence

$$\int_{0}^{\infty} e^{-x^2} (e^{2ax} + e^{-2ax}) dx = e^{a^2} \sqrt{\pi}.$$

In this equation, replace a by $\alpha\sqrt{-1}$; then, since

$$e^{2ax} + e^{-2ax} = e^{-2\alpha x\sqrt{-1}} + e^{2\alpha x\sqrt{-1}} = 2 \cos. 2\alpha x \text{ (Art. 73)},$$

we have

$$\int_{0}^{\infty} e^{-x^2} \cos. 2\alpha x dx = \frac{1}{2} e^{-\alpha^2} \sqrt{\pi}.$$

This example is another instance in which the value of a definite integral is found by passing from real to imaginary quantities.

278. Another process by which $\int_0^\infty e^{-x^2}\cos. 2\alpha x dx$ may be found consists in differentiating with respect to α and subsequent integration: thus, put

$$u = \int_0^\infty e^{-x^2}\cos. 2\alpha x dx;$$

then

$$\frac{du}{d\alpha} = -\int_0^\infty \sin. 2\alpha x e^{-x^2} 2x dx = \int_0^\infty \sin. 2\alpha x, d, e^{-x^2}.$$

Integrating by parts, and observing, that, at the limits, $\sin. 2\alpha x e^{-x^2}$ is zero, we have

$$\frac{du}{d\alpha} = -\int_0^\infty e^{-x^2}\cos. 2\alpha x 2\alpha dx = -2\alpha u:$$

$$\therefore \qquad \frac{\frac{du}{d\alpha}}{u} = -2\alpha.$$

But, regarding u as a function of α, we have

$$\frac{d}{d\alpha} lu = \frac{\frac{du}{d\alpha}}{u}: \ \therefore \ \frac{d}{d\alpha} lu = -2\alpha.$$

Integrating with respect to α, we get

$$lu = -\alpha^2 + C: \ \therefore \ u = e^{-\alpha^2 + c} = Ce^{-\alpha^2}$$

by making $e^c = C$. To determine C, make $\alpha = 0$; then

$$u = \int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi} = C:$$

therefore

$$\int_0^\infty e^{-x^2}\cos. 2\alpha x dx = \frac{1}{2} e^{-\alpha^2}\sqrt{\pi}.$$

SECTION X.

ELLIPTIC FUNCTIONS.

279. ***Elliptic Functions*** or ***Elliptic Integrals*** is the name given to the following integrals: —

First order. $$\int_0^\theta \frac{d\theta}{\sqrt{1-c^2\sin.^2\theta}} = F(c,\theta).$$

Second order. $$\int_0^\theta \sqrt{1-c^2\sin.^2\theta}\,d\theta = E(c,\theta).$$

Third order. $$\int_0^\theta \frac{d\theta}{(1+a\sin.^2\theta)\sqrt{1-c^2\sin.^2\theta}} = \Pi(c,a,\theta).$$

The constant c is called the *modulus* of the function, and is supposed less than unity; the constant a, which appears in the third function, is called the *parameter;* and the variable θ is called the *amplitude* of the function. The function is said to be complete when the limits of the amplitude are 0 and $\frac{\pi}{2}$. The integral of the second order expresses the length of the arc of an ellipse estimated from the vertex of the conjugate axis (Art. 240); the semi-transverse axis being unity, and the eccentricity of the ellipse the modulus of the integral. From this fact, and from the relations which exist between the several functions, the term *elliptic functions* has been derived. Our limits permit us to investigate but a few propositions relating to such functions.

280. Putting x for $\sin.\theta$, the integral of the first order becomes $$\int_0^x \frac{dx}{\sqrt{1-x^2}\sqrt{1-c^2x^2}}.$$

In like manner, for another value of x denoted by x_1, we have

$$\int_0^{x_1} \frac{dx_1}{\sqrt{1-x_1^2}\sqrt{1-c^2x_1^2}}.$$

Now assume the relation

$$\frac{dx}{\sqrt{1-x^2}\sqrt{1-c^2x^2}} + \frac{dx_1}{\sqrt{1-x_1^2}\sqrt{1-c^2x_1^2}} = 0 \quad (1).$$

Multiply through by the product of the denominators, divide by $1-c^2x_1^2$, and integrate; then

$$\int \frac{\sqrt{1-x_1^2}\sqrt{1-c^2x_1^2}}{1-c^2x^2x_1^2}dx + \int \frac{\sqrt{1-x^2}\sqrt{1-c^2x^2}}{1-c^2x^2x_1^2}dx = \text{constant.}$$

Integrating the first term by parts, we get

$$\int \frac{\sqrt{1-x_1^2}\sqrt{1-c^2x_1^2}}{1-c^2x^2x_1^2}dx = \frac{x\sqrt{1-x_1^2}\sqrt{1-c^2x_1^2}}{1-c^2x^2x_1^2}$$
$$+\int xx_1\frac{(1+c^2)(1+c^2x^2x_1^2)-2cx^2-2c^2x_1^2}{(1-c^2x^2x_1^2)^2}\cdot\frac{dx_1}{\sqrt{1-x_1^2}\sqrt{1-c^2x_1^2}}.$$
$$-2c^2\int\frac{x^2x_1^2}{1-c^2x^2x_1^2}\sqrt{1-x_1^2}\sqrt{1-c^2x_1^2}\,dx.$$

In this result, interchanging x and x_1, we have the second term. Adding results, observing that by (1) the terms of the sum which are under the sign $\int$ reduce to zero, we find

$$\frac{x\sqrt{1-x_1^2}\sqrt{1-c^2x_1^2}+x_1\sqrt{1-x^2}\sqrt{1-c^2x^2}}{1-c^2x^2x_1^2} = \text{const.} \quad (2).$$

Eq. 1 expresses the condition that the variables x and x_1 are so related that the sum of the integrals

$$\int_0^x \frac{dx}{\sqrt{1-x^2}\sqrt{1-c^2x^2}}, \quad \int_0^{x_1}\frac{dx_1}{\sqrt{1-x_1^2}\sqrt{1-c^2x_1^2}},$$

shall be constant.

Put $\int_0^x \frac{dx}{\sqrt{1-x^2}\sqrt{1-c^2x^2}} = \alpha,\ x = S(\alpha),$

$$\sqrt{1-x^2} = C(\alpha),\ \sqrt{1-c^2x^2} = R(\alpha);$$

also

$$\int_0^{x_1} \frac{dx}{\sqrt{1-x_1^2}\sqrt{1-c^2x_1^2}} = \beta,\ x_1 = S(\beta),$$

$$\sqrt{1-x_1^2} = C(\beta),\ \sqrt{1-c^2x_1^2} = R(\beta).$$

Then, by Eq. 1, we have

$$d\alpha + d\beta = 0:$$

$$\therefore \qquad \alpha + \beta = \text{constant} = \gamma.$$

It is also seen from (1) that the constant γ is the value of x_1 when $x = 0$; and, further, when $\alpha = 0$, we have

$$x = 0,\ \beta = \gamma,\ x_1 = S(\gamma) = S(\alpha + \beta):$$

therefore, by making the proper substitutions in (2), it becomes

$$S(\alpha+\beta) = \frac{S(\alpha)C(\beta)R(\beta) + S(\beta)C(\alpha)R(\alpha)}{1 - c^2\left\{S(\alpha)\right\}^2\left\{S(\beta)\right\}^2},$$

which is the fundamental formula as given by Euler in the theory of elliptic functions.

281. Suppose the variables θ, θ_1, to be connected by the equation

$$\int_0^\theta \frac{d\theta}{\sqrt{1-c^2\sin.^2\theta}} + \int_0^{\theta_1} \frac{d\theta_1}{\sqrt{1-c^2\sin.^2\theta_1}} = \int_0^\mu \frac{d\theta}{\sqrt{1-c^2\sin.^2\theta}} \quad (1),$$

or

$$F(c,\theta) + F(c,\theta_1) = F(c,\mu),$$

in which μ is a constant. If θ, θ_1, be regarded as functions of a third variable t, and (1) be differentiated with respect to the latter variable, we have

$$\frac{\frac{d\theta}{dt}}{\sqrt{1-c^2\sin.^2\theta}} + \frac{\frac{d\theta_1}{dt}}{\sqrt{1-c^2\sin.^2\theta_1}} = 0 \quad (2).$$

Since the new variable t is arbitrary, let us assume

$$\frac{d\theta}{dt} = \sqrt{(1 - c^2 \sin.^2\theta)} \quad (3);$$

whence, from (2),

$$\frac{d\theta_1}{dt} = -\sqrt{(1 - c^2 \sin.^2\theta_1)} \quad (4).$$

Squaring (3) and (4), and differentiating, we get

$$\frac{d^2\theta}{dt^2} = -c^2 \sin.\theta \cos.\theta, \quad \frac{d^2\theta_1}{dt^2} = -c^2 \sin.\theta_1 \cos.\theta_1:$$

$$\therefore \frac{d^2\theta}{dt^2} \pm \frac{d^2\theta_1}{dt^2} = -c^2(\sin.\theta \cos.\theta \pm \sin.\theta_1 \cos.\theta_1),$$

$$\text{or } \frac{d^2}{dt^2}(\theta \pm \theta_1) = -\frac{c^2}{2}(\sin.2\theta \pm \sin.2\theta_1) \quad (5).$$

Put $\theta + \theta_1 = \varphi$, and $\theta - \theta_1 = \psi$; then

$$2\theta = \varphi + \psi, \quad 2\theta_1 = \varphi - \psi,$$

$$\sin.2\theta = \sin.\varphi \cos.\psi + \cos.\varphi \sin.\psi,$$

$$\sin.2\theta_1 = \sin.\varphi \cos.\psi - \cos.\varphi \sin.\psi:$$

therefore, from (5), we have

$$\frac{d^2\varphi}{dt^2} = -c^2 \sin.\varphi \cos.\psi, \quad \frac{d^2\psi}{dt^2} = -c^2 \sin.\psi \cos.\varphi.$$

We also have

$$\frac{d\varphi}{dt}\frac{d\psi}{dt} = \left(\frac{d\theta}{dt}\right)^2 - \left(\frac{d\theta_1}{dt}\right)^2 = c^2\left(\frac{1 - \cos.2\theta_1}{2} - \frac{1 - \cos.2\theta}{2}\right)$$

$$= c^2\left(\frac{\cos.2\theta}{2} - \frac{\cos.2\theta_1}{2}\right) = -c^2 \sin.\varphi \sin.\psi:$$

$$\therefore \quad \frac{\frac{d^2\varphi}{dt^2}}{\frac{d\varphi}{dt}\frac{d\psi}{dt}} = \cot.\psi, \quad \frac{\frac{d^2\psi}{dt^2}}{\frac{d\varphi}{dt}\frac{d\psi}{dt}} = \cot.\varphi.$$

But $\frac{d}{dt} l \sin. \varphi = \cot. \varphi \frac{d\varphi}{dt}, \; \frac{d}{dt} l \frac{d\psi}{dt} = \frac{\frac{d^2\psi}{dt^2}}{\frac{d\psi}{dt}}$:

$$\therefore \quad \frac{d}{dt}\left(l \frac{d\varphi}{dt}\right) = \frac{d}{dt} l \sin. \psi, \; \frac{d}{dt}\left(l \frac{d\psi}{dt}\right) = \frac{d}{dt} l \sin. \varphi.$$

Whence

$$l \frac{d\varphi}{dt} = l \sin. \psi + C, \; l \frac{d\psi}{dt} = l \sin. \varphi + C_1;$$

or by putting $C = lA$, $C_1 = lA_1$, and passing from logarithms to numbers,

$$\frac{d\varphi}{dt} = A \sin. \psi, \; \frac{d\psi}{dt} = A_1 \sin. \varphi \quad (6):$$

$$\therefore \quad A \sin. \psi \frac{d\psi}{dt} = A_1 \sin. \varphi \frac{d\varphi}{dt};$$

$$\therefore \quad A \cos. \psi = A_1 \cos. \varphi + C \quad (7).$$

From Eq. 1 wo see that $F(c, \theta) = F(c, \mu)$ when $\theta_1 = 0$: therefore we then have $\theta = \mu = \varphi = \psi$, and (7) then becomes $(A - A_1) \cos. \mu = C$; and therefore

$$A \cos. (\theta - \theta_1) - A_1 \cos. (\theta + \theta_1) = (A - A_1) \cos. \mu;$$

whence, by developing $\cos. (\theta - \theta_1)$, $\cos. (\theta + \theta_1)$, and reducing,

$$(A - A_1) \cos. \theta \cos. \theta_1 + (A + A_1) \sin. \theta \sin. \theta_1 = (A - A_1) \cos. \mu \quad (8).$$

Now,

$$\frac{d\varphi}{dt} = \frac{d\theta}{dt} + \frac{d\theta_1}{dt} = \sqrt{(1 - c^2 \sin.^2 \theta)} - \sqrt{(1 - c^2 \sin.^2 \theta_1)},$$

$$\frac{d\psi}{dt} = \sqrt{(1 - c^2 \sin.^2 \theta)} + \sqrt{(1 - c^2 \sin.^2 \theta_1)}.$$

Substitute these values in (6), and make $\theta_1 = 0$; then

$$\sqrt{(1 - c^2 \sin.^2 \mu)} - 1 = A \sin. \mu,$$

$$\sqrt{(1 - c^2 \sin.^2 \mu)} + 1 = A_1 \sin. \mu.$$

From these equations, getting the values of $A + A_1$, $A - A_1$, and substituting in (8), we get, finally,

$$\cos.\theta \cos.\theta_1 - \sin.\theta \sin.\theta_1 \sqrt{(1 - c^2 \sin.^2\mu)} = \cos.\mu \quad (9).$$

This relation, by an easy transformation, may be made to take the form

$$\cos.\theta = \cos.\theta_1 \cos.\mu + \sin.\theta_1 \sin.\mu \sqrt{(1 - c^2 \sin.^2\theta)} \quad (10).$$

Eqs. (9) and (10) express the connection which exists between the variables in two elliptic functions of the first order which have a common modulus.

282. Let $F(c, \theta)$, $F(c, \theta_1)$, be two elliptic functions in which c, c_1, and θ, θ_1, are connected by the equations

$$c_1^2 = \frac{4c}{(1+c)^2} \quad (1), \qquad \tan.\theta = \frac{\sin.2\theta_1}{c + \cos.2\theta_1} \quad (2).$$

It is proposed to prove that

$$F(c, \theta) = \frac{2}{1+c} F(c_1, \theta_1).$$

Differentiate Eq. 2, regarding θ_1 as the independent variable; then

$$\frac{1}{\cos.^2\theta} \frac{d\theta}{d\theta_1} = \frac{2(1 + c\cos.2\theta_1)}{(c + \cos.2\theta_1)^2}.$$

From (2) we also get

$$\cos.^2\theta = \frac{(c + \cos.2\theta_1)^2}{1 + 2c\cos.2\theta_1 + c^2}:$$

$$\therefore \quad \frac{d\theta}{d\theta_1} = \frac{2(1 + c\cos.2\theta_1)}{1 + 2c\cos.2\theta_1 + c^2}.$$

Also, from the same equation, we get

$$1 - c^2 \sin.^2\theta = 1 - \frac{c^2 \sin.^2 2\theta_1}{1 + 2c\cos.2\theta_1 + c^2}$$

$$= \frac{1 + 2c\cos.2\theta_1 + c^2 \cos.^2 2\theta_1}{1 + 2c\cos.2\theta_1 + c^2}:$$

$$\therefore \int \frac{d\theta}{\sqrt{(1 - c^2 \sin.^2 \theta)}}$$

$$= \int \frac{2(1 + c\cos. 2\theta_1)}{1 + 2c\cos. 2\theta_1 + c^2} \; \frac{\sqrt{(1 + 2c\cos. 2\theta_1 + c^2)}}{1 + c\cos. 2\theta_1} d\theta_1$$

$$= 2\int \frac{d\theta_1}{\sqrt{(1 + 2c\cos. 2\theta_1 + c^2)}}$$

$$= \frac{2}{1+c} \int \frac{d\theta_1}{\sqrt{\left(1 - \frac{4c}{(1+c)^2} \sin.^2 \theta_1\right)}}.$$

But the last integral, when $\frac{4c}{(1+c)^2} = c_1^2$, becomes

$$\frac{2}{1+c} \int \frac{d\theta_1}{\sqrt{(1 - c_1^2 \sin.^2 \theta_1)}} = \frac{2}{1+c} F(c_1, \theta_1):$$

$$\therefore \qquad F(c, \theta) = \frac{2}{1+c} F(c_1, \theta_1).$$

If we suppose $\theta_1 = \frac{\pi}{2}$, $\theta = \pi$, then

$$\frac{2}{1+c} F\left(c_1, \frac{\pi}{2}\right) = F(c, \pi) = 2F\left(c, \frac{\pi}{2}\right).$$

283. Having shown (Art. 281) that there exists, between the variables of two elliptic functions of the first order having a common modulus, the relation

$$\cos. \theta \cos. \theta_1 - \sin. \theta \sin. \theta_1 \sqrt{(1 - c^2 \sin. \mu)} = \cos. \mu \quad (1),$$

then, between the corresponding functions of the second order, there exists the relation

$$E(c, \theta) + E(c, \theta_1) - E(c, \mu) = c^2 \sin. \theta \sin. \theta_1 \sin. \mu.$$

From the equation between the amplitudes θ, θ_1, θ_1, may be considered as a function of θ; that is, we may assume

$$E(c, \theta) + E(c, \theta_1) - E(c, \mu) = f(\theta),$$

and differentiate, thus getting

$$\sqrt{(1 - c^2 \sin.^2 \theta)} + \sqrt{(1 - c^4 \sin.^2 \theta_1)} \frac{d\theta_1}{d\theta} = f'(\theta).$$

By Eq. 10, Art. 281, the first member of this equation may be put under the form

$$\frac{\cos.\theta - \cos.\theta_1 \cos.\mu}{\sin.\theta_1 \sin.\mu} + \frac{\cos.\theta_1 - \cos.\theta \cos.\mu}{\sin.\theta \sin.\mu} \frac{d\theta_1}{d\theta}$$

$$= \frac{d(\sin.^2 \theta + \sin.^2 \theta_1 + 2 \cos.\theta \cos.\theta_1 \cos.\mu}{d\theta} \frac{1}{2 \sin.\theta \sin.\theta_1 \sin.\mu}.$$

But putting Eq. 1 under the form

$$\cos.\theta \cos.\theta_1 - \cos.\mu = \sqrt{(1 - c^2 \sin.^2 \mu)} \sin.\theta \sin.\theta_1,$$

and squaring, we get

$$\cos.^{2\theta} + \cos.^{2\theta_1} + \cos.^2 \mu - 2 \cos.\theta \cos.\theta_1 \cos.\mu$$
$$= (1 - c^2 \sin.^2 \mu) \sin.^2 \theta \sin.^2 \theta_1.$$

Adding $\cos.^2 \theta_1 \cos.^2 \mu$ to both sides of this equation, transposing, and reducing by the relation $\cos.^2 = 1 - \sin.^2$, we find

$$\sin.^2 \theta + \sin.^2 \theta_1 + 2 \cos.\theta \cos.\theta_1 \cos.\mu$$
$$= 1 + \cos.^2 \mu + c^2 \sin.^2 \theta \sin.^2 \theta_1 \sin.^2 \mu,$$

$$\frac{\frac{d}{d\theta}(1 + \cos.^2 \mu + c^2 \sin.^2 \theta \sin.^2 \theta_1 \sin.^2 \mu)}{2 \sin.\theta \sin.\theta_1 \sin.\mu}$$
$$= c^2 \sin.\mu \frac{d(\sin.\theta \sin.\theta_1)}{d\theta}:$$

$$\therefore \qquad f^{-1}(\theta) = c^2 \sin.\mu \frac{d(\sin.\theta \sin.\theta_1)}{d\theta};$$

and therefore, by integration,

$$f(\theta) = c^2 \sin.\theta \sin.\theta_1 \sin.\mu.$$

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